6

DISCOUNTED CASH FLOW VALUATION

THE SIGNING OF BIG-NAME ATHLETES is often accompanied by great fanfare, but the numbers are often misleading. For example, in 2006, catcher Ramon Hernandez joined the Baltimore Orioles, signing a contract with a reported value of $27.5 million. Not bad, especially for someone who makes a living using the "tools of ignorance" (jock jargon for catcher's equipment). Another example is the contract signed by wide receiver Chad Johnson of the Cincinnati Bengals, which had a stated value of about $35.5 million.

A closer look at the numbers shows that both Ramon and Chad did pretty well, but nothing like the quoted figures. Using Chad's contract as an example, while the value was reported to be $35.5 million, this amount was actually payable over several years. It consisted of $8.25 million in the first year plus $27.25 million in future salary and bonuses paid in the years 2007 through 2011. Ramon's payments were similarly spread over time. Because both contracts called for payments that are made at future dates, we must consider the time value of money, which means neither player received the quoted amounts. How much did they really get? This chapter gives you the "tools of knowledge" to answer this question.

In our previous chapter, we covered the basics of discounted cash flow valuation. However, so far, we have dealt with only single cash flows. In reality, most investments have multiple cash flows. For example, if Sears is thinking of opening a new department store, there will be a large cash outlay in the beginning and then cash inflows for many years. In this chapter, we begin to explore how to value such investments.

When you finish this chapter, you should have some very practical skills. For example, you will know how to calculate your own car payments or student loan payments. You will also be able to determine how long it will take to pay off a credit card if you make the minimum payment each month (a practice we do not recommend). We will show you how to compare interest rates to determine which are the highest and which are the lowest, and we will also show you how interest rates can be quoted in different—and at times deceptive—ways.
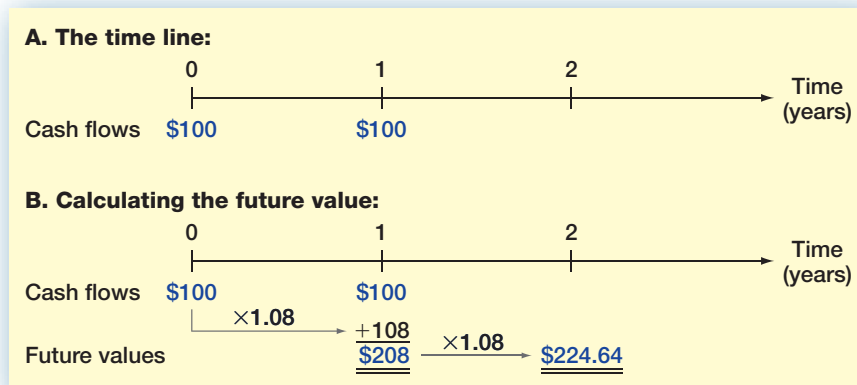
A. The time line:

	0	1	2	
	├─────────┼─────────┼─────────→			Time (years)
Cash flows	$100	$100		

B. Calculating the future value:

	0	1	2	
	├─────────┼─────────┼─────────→			Time (years)
Cash flows	$100	$100		
	└──×1.08──→ +108			
Future values		$208 ──×1.08──→	$224.64	

Future and Present Values of Multiple Cash Flows

6.1

Thus far, we have restricted our attention to either the future value of a lump sum present amount or the present value of some single future cash flow. In this section, we begin to study ways to value multiple cash flows. We start with future value.

FUTURE VALUE WITH MULTIPLE CASH FLOWS

Suppose you deposit $100 today in an account paying 8 percent. In one year, you will deposit another $100. How much will you have in two years? This particular problem is relatively easy. At the end of the first year, you will have $108 plus the second $100 you deposit, for a total of $208. You leave this $208 on deposit at 8 percent for another year. At the end of this second year, it is worth:

$$\$208 \times 1.08 = \$224.64$$

Figure 6.1 is a *time line* that illustrates the process of calculating the future value of these two $100 deposits. Figures such as this are useful for solving complicated problems. Almost anytime you are having trouble with a present or future value problem, drawing a time line will help you see what is happening.

In the first part of Figure 6.1, we show the cash flows on the time line. The most important thing is that we write them down where they actually occur. Here, the first cash flow occurs today, which we label as time 0. We therefore put $100 at time 0 on the time line. The second $100 cash flow occurs one year from today, so we write it down at the point labeled as time 1. In the second part of Figure 6.1, we calculate the future values one period at a time to come up with the final $224.64.

Saving Up Revisited	**EXAMPLE 6.1**

You think you will be able to deposit $4,000 at the end of each of the next three years in a bank account paying 8 percent interest. You currently have $7,000 in the account. How much will you have in three years? In four years?

At the end of the first year, you will have:

$$\$7,000 \times 1.08 + 4,000 = \$11,560$$

(continued)

At the end of the second year, you will have:

$11,560 \times 1.08 + 4,000 = \$16,484.80$

Repeating this for the third year gives:

$16,484.80 \times 1.08 + 4,000 = \$21,803.58$

Therefore, you will have $21,803.58 in three years. If you leave this on deposit for one more year (and don't add to it), at the end of the fourth year, you'll have:

$21,803.58 \times 1.08 = \$23,547.87$

When we calculated the future value of the two $100 deposits, we simply calculated the balance as of the beginning of each year and then rolled that amount forward to the next year. We could have done it another, quicker way. The first $100 is on deposit for two years at 8 percent, so its future value is:

$100 \times 1.08^2 = \$100 \times 1.1664 = \$116.64$

The second $100 is on deposit for one year at 8 percent, and its future value is thus:

$100 \times 1.08 = \$108$

The total future value, as we previously calculated, is equal to the sum of these two future values:

$116.64 + 108 = \$224.64$

Based on this example, there are two ways to calculate future values for multiple cash flows: (1) Compound the accumulated balance forward one year at a time or (2) calculate the future value of each cash flow first and then add them up. Both give the same answer, so you can do it either way.

To illustrate the two different ways of calculating future values, consider the future value of $2,000 invested at the end of each of the next five years. The current balance is zero, and the rate is 10 percent. We first draw a time line, as shown in Figure 6.2.

On the time line, notice that nothing happens until the end of the first year, when we make the first $2,000 investment. This first $2,000 earns interest for the next four (not five) years. Also notice that the last $2,000 is invested at the end of the fifth year, so it earns no interest at all.

Figure 6.3 illustrates the calculations involved if we compound the investment one period at a time. As illustrated, the future value is $12,210.20.

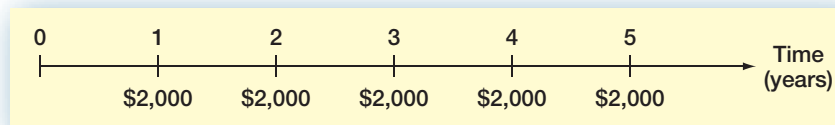**FIGURE 6.2**

Time Line for $2,000 per Year for Five Years

FIGURE 6.3 Future Value Calculated by Compounding Forward One Period at a Time
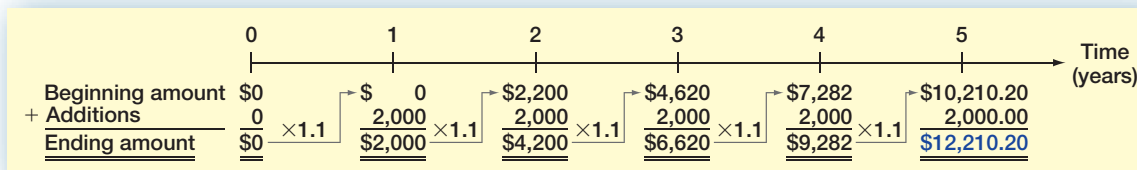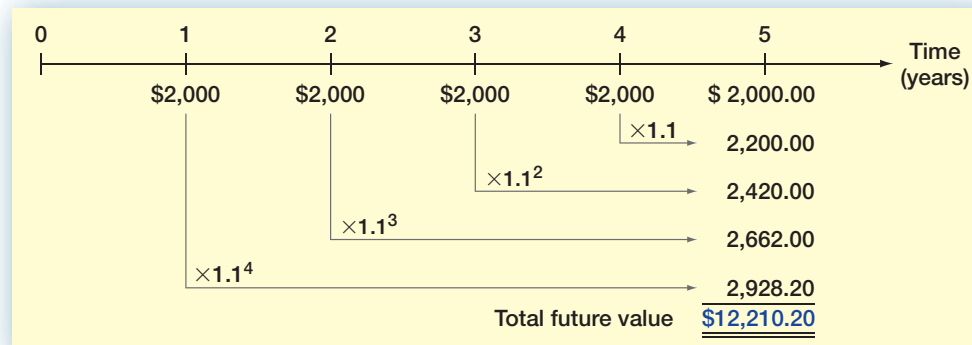
FIGURE 6.4 Future Value Calculated by Compounding Each Cash Flow Separately

Figure 6.4 goes through the same calculations, but the second technique is used. Naturally, the answer is the same.

Saving Up Once Again	**EXAMPLE 6.2**

If you deposit $100 in one year, $200 in two years, and $300 in three years, how much will you have in three years? How much of this is interest? How much will you have in five years if you don't add additional amounts? Assume a 7 percent interest rate throughout.

We will calculate the future value of each amount in three years. Notice that the $100 earns interest for two years, and the $200 earns interest for one year. The final $300 earns no interest. The future values are thus:

$$\begin{aligned}
\$100 \times 1.07^2 &= \$114.49 \\
\$200 \times 1.07 &= 214.00 \\
+\$300 &= \underline{300.00} \\
\text{Total future value} &= \underline{\$628.49}
\end{aligned}$$

The total future value is thus $628.49. The total interest is:

$$\$628.49 - (100 + 200 + 300) = \$28.49$$

How much will you have in five years? We know that you will have $628.49 in three years. If you leave that in for two more years, it will grow to:

$$\$628.49 \times 1.07^2 = \$628.49 \times 1.1449 = \$719.56$$

Notice that we could have calculated the future value of each amount separately. Once again, be careful about the lengths of time. As we previously calculated, the first $100 earns interest for only four years, the second deposit earns three years' interest, and the last earns two years' interest:

$$\begin{aligned}
\$100 \times 1.07^4 &= \$100 \times 1.3108 = \$131.08 \\
\$200 \times 1.07^3 &= \$200 \times 1.2250 = 245.01 \\
+\$300 \times 1.07^2 &= \$300 \times 1.1449 = \underline{343.47} \\
\text{Total future value} &= \underline{\$719.56}
\end{aligned}$$

PRESENT VALUE WITH MULTIPLE CASH FLOWS

We often need to determine the present value of a series of future cash flows. As with future values, there are two ways we can do it. We can either discount back one period at a time, or we can just calculate the present values individually and add them up.

Suppose you need $1,000 in one year and $2,000 more in two years. If you can earn 9 percent on your money, how much do you have to put up today to exactly cover these amounts in the future? In other words, what is the present value of the two cash flows at 9 percent?

The present value of $2,000 in two years at 9 percent is:

$$\$2,000/1.09^2 = \$1,683.36$$

The present value of $1,000 in one year is:

$$\$1,000/1.09 = \$917.43$$

Therefore, the total present value is:

$$\$1,683.36 + 917.43 = \$2,600.79$$

To see why $2,600.79 is the right answer, we can check to see that after the $2,000 is paid out in two years, there is no money left. If we invest $2,600.79 for one year at 9 percent, we will have:

$$\$2,600.79 \times 1.09 = \$2,834.86$$

We take out $1,000, leaving $1,834.86. This amount earns 9 percent for another year, leaving us with:

$$\$1,834.86 \times 1.09 = \$2,000$$

This is just as we planned. As this example illustrates, the present value of a series of future cash flows is simply the amount you would need today to exactly duplicate those future cash flows (for a given discount rate).

An alternative way of calculating present values for multiple future cash flows is to discount back to the present, one period at a time. To illustrate, suppose we had an investment that was going to pay $1,000 at the end of every year for the next five years. To find the present value, we could discount each $1,000 back to the present separately and then add them up. Figure 6.5 illustrates this approach for a 6 percent discount rate; as shown, the answer is $4,212.37 (ignoring a small rounding error).

FIGURE 6.5

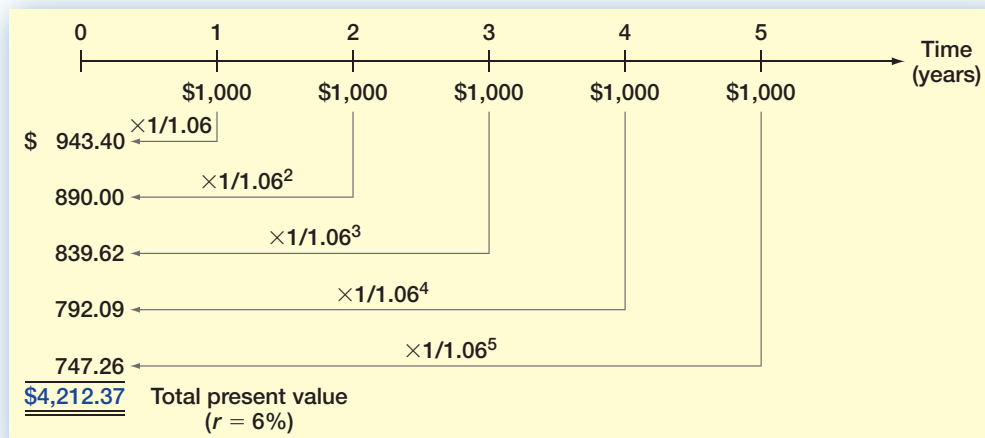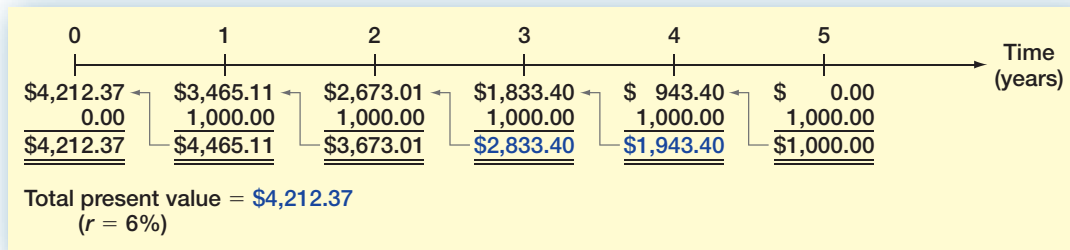Present Value Calculated by Discounting Each Cash Flow Separately

FIGURE 6.6   Present Value Calculated by Discounting Back One Period at a Time

Alternatively, we could discount the last cash flow back one period and add it to the next-to-the-last cash flow:

$$(\$1,000/1.06) + 1,000 = \$943.40 + 1,000 = \$1,943.40$$

We could then discount this amount back one period and add it to the year 3 cash flow:

$$(\$1,943.40/1.06) + 1,000 = \$1,833.40 + 1,000 = \$2,833.40$$

This process could be repeated as necessary. Figure 6.6 illustrates this approach and the remaining calculations.

How Much Is It Worth?	**EXAMPLE 6.3**

You are offered an investment that will pay you $200 in one year, $400 the next year, $600 the next year, and $800 at the end of the fourth year. You can earn 12 percent on very similar investments. What is the most you should pay for this one?

We need to calculate the present value of these cash flows at 12 percent. Taking them one at a time gives:

$$\$200 \times 1/1.12^1 = \$200/1.1200 = \$\ \ 178.57$$
$$\$400 \times 1/1.12^2 = \$400/1.2544 = \ \ \ \ 318.88$$
$$\$600 \times 1/1.12^3 = \$600/1.4049 = \ \ \ \ 427.07$$
$$+\$800 \times 1/1.12^4 = \$800/1.5735 = \ \ \underline{\ \ \ 508.41}$$
$$\text{Total present value} = \underline{\$1,432.93}$$

If you can earn 12 percent on your money, then you can duplicate this investment's cash flows for $1,432.93, so this is the most you should be willing to pay.

How Much Is It Worth? Part 2	**EXAMPLE 6.4**

You are offered an investment that will make three $5,000 payments. The first payment will occur four years from today. The second will occur in five years, and the third will follow in six years. If you can earn 11 percent, what is the most this investment is worth today? What is the future value of the cash flows?

We will answer the questions in reverse order to illustrate a point. The future value of the cash flows in six years is:

$$(\$5,000 \times 1.11^2) + (5,000 \times 1.11) + 5,000 = \$6,160.50 + 5,550 + 5,000$$
$$= \$16,710.50$$

(continued)

The present value must be:

$16,710.50/1.11^6 = $8,934.12$

Let's check this. Taking them one at a time, the PVs of the cash flows are:

$$\$5,000 \times 1/1.11^6 = \$5,000/1.8704 = \$2,673.20$$
$$\$5,000 \times 1/1.11^5 = \$5,000/1.6851 = \ 2,967.26$$
$$+\$5,000 \times 1/1.11^4 = \$5,000/1.5181 = \ \underline{3,293.65}$$
$$\text{Total present value} = \underline{\$8,934.12}$$

This is as we previously calculated. The point we want to make is that we can calculate present and future values in any order and convert between them using whatever way seems most convenient. The answers will always be the same as long as we stick with the same discount rate and are careful to keep track of the right number of periods.

CALCULATOR HINTS

How to Calculate Present Values with Multiple Future Cash Flows Using a Financial Calculator

To calculate the present value of multiple cash flows with a financial calculator, we will simply discount the individual cash flows one at a time using the same technique we used in our previous chapter, so this is not really new. However, we can show you a shortcut. We will use the numbers in Example 6.3 to illustrate.

To begin, of course we first remember to clear out the calculator! Next, from Example 6.3, the first cash flow is $200 to be received in one year and the discount rate is 12 percent, so we do the following:

Enter	1	12		200	
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for				−178.57	

Now, you can write down this answer to save it, but that's inefficient. All calculators have a memory where you can store numbers. Why not just save it there? Doing so cuts way down on mistakes because you don't have to write down and/or rekey numbers, and it's much faster.

Next we value the second cash flow. We need to change N to 2 and FV to 400. As long as we haven't changed anything else, we don't have to reenter I/Y or clear out the calculator, so we have:

Enter	2			400	
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for				−318.88	

You save this number by adding it to the one you saved in our first calculation, and so on for the remaining two calculations.

As we will see in a later chapter, some financial calculators will let you enter all of the future cash flows at once, but we'll discuss that subject when we get to it.

SPREADSHEET STRATEGIES

How to Calculate Present Values with Multiple Future Cash Flows Using a Spreadsheet

Just as we did in our previous chapter, we can set up a basic spreadsheet to calculate the present values of the individual cash flows as follows. Notice that we have simply calculated the present values one at a time and added them up:

	A	B	C	D	E
1					
2			Using a spreadsheet to value multiple future cash flows		
3					
4	What is the present value of $200 in one year, $400 the next year, $600 the next year, and				
5	$800 the last year if the discount rate is 12 percent?				
6					
7	Rate:	0.12			
8					
9	Year	Cash flows	Present values	Formula used	
10	1	$200	$178.57	=PV($B$7,A10,0,−B10)	
11	2	$400	$318.88	=PV($B$7,A11,0,−B11)	
12	3	$600	$427.07	=PV($B$7,A12,0,−B12)	
13	4	$800	$508.41	=PV($B$7,A13,0,−B13)	
14					
15		Total PV:	**$1,432.93**	=SUM(C10:C13)	
16					
17	Notice the negative signs inserted in the PV formulas. These just make the present values have				
18	positive signs. Also, the discount rate in cell B7 is entered as $B$7 (an "absolute" reference)				
19	because it is used over and over. We could have just entered ".12" instead, but our approach is more				
20	flexible.				
21					
22					

A NOTE ABOUT CASH FLOW TIMING

In working present and future value problems, cash flow timing is critically important. In almost all such calculations, it is implicitly assumed that the cash flows occur at the *end* of each period. In fact, all the formulas we have discussed, all the numbers in a standard present value or future value table, and (very important) all the preset (or default) settings on a financial calculator assume that cash flows occur at the end of each period. Unless you are explicitly told otherwise, you should always assume that this is what is meant.

As a quick illustration of this point, suppose you are told that a three-year investment has a first-year cash flow of $100, a second-year cash flow of $200, and a third-year cash flow of $300. You are asked to draw a time line. Without further information, you should always assume that the time line looks like this:

```
0          1          2          3
+----------+----------+----------+--------->
         $100       $200       $300
```
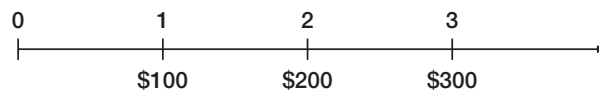
On our time line, notice how the first cash flow occurs at the end of the first period, the second at the end of the second period, and the third at the end of the third period.

We will close this section by answering the question we posed at the beginning of the chapter concerning Chad Johnson's NFL contract. Recall that the contract called for $8.25 million in the first year. The remaining $27.25 million was to be paid as $7.75 million in 2007, $3.25 million in 2008, $4.75 million in 2009, $5.25 million in 2010, and $6.25 million in 2011. If 12 percent is the appropriate interest rate, what kind of deal did the Bengals' wide receiver catch?

To answer, we can calculate the present value by discounting each year's salary back to the present as follows (notice we assume that all the payments are made at year-end):

Year 1 (2006): $8,250,000 \times 1/1.12^1 = \$7,366,071.43$

Year 2 (2007): $7,750,000 \times 1/1.12^2 = \$6,178,252.55$

Year 3 (2008): $3,250,000 \times 1/1.12^3 = \$2,313,285.81$

          .

          .

Year 6 (2011): $6,250,000 \times 1/1.12^6 = \$3,166,444.51$

If you fill in the missing rows and then add (do it for practice), you will see that Johnson's contract had a present value of about $25 million, or about 70 percent of the stated $35.5 million value.

Concept Questions

6.1a Describe how to calculate the future value of a series of cash flows.

6.1b Describe how to calculate the present value of a series of cash flows.

6.1c Unless we are explicitly told otherwise, what do we always assume about the timing of cash flows in present and future value problems?

6.2 Valuing Level Cash Flows: Annuities and Perpetuities

We will frequently encounter situations in which we have multiple cash flows that are all the same amount. For example, a common type of loan repayment plan calls for the borrower to repay the loan by making a series of equal payments over some length of time. Almost all consumer loans (such as car loans) and home mortgages feature equal payments, usually made each month.

More generally, a series of constant or level cash flows that occur at the end of each period for some fixed number of periods is called an ordinary **annuity**; more correctly, the cash flows are said to be in *ordinary annuity form*. Annuities appear frequently in financial arrangements, and there are some useful shortcuts for determining their values. We consider these next.

annuity
A level stream of cash flows for a fixed period of time.

PRESENT VALUE FOR ANNUITY CASH FLOWS

Suppose we were examining an asset that promised to pay $500 at the end of each of the next three years. The cash flows from this asset are in the form of a three-year, $500 annuity. If we wanted to earn 10 percent on our money, how much would we offer for this annuity?

From the previous section, we know that we can discount each of these $500 payments back to the present at 10 percent to determine the total present value:

$$Present\ value = (\$500/1.1^1) + (500/1.1^2) + (500/1.1^3)$$
$$= (\$500/1.1) + (500/1.21) + (500/1.331)$$
$$= \$454.55 + 413.22 + 375.66$$
$$= \$1,243.43$$

This approach works just fine. However, we will often encounter situations in which the number of cash flows is quite large. For example, a typical home mortgage calls for monthly payments over 30 years, for a total of 360 payments. If we were trying to determine the present value of those payments, it would be useful to have a shortcut.

Because the cash flows of an annuity are all the same, we can come up with a handy variation on the basic present value equation. The present value of an annuity of C dollars per period for t periods when the rate of return or interest rate is r is given by:

$$Annuity\ present\ value = C \times \left(\frac{1 - Present\ value\ factor}{r} \right)$$

$$= C \times \left\{ \frac{1 - [1/(1 + r)^t]}{r} \right\}$$ [6.1]

The term in parentheses on the first line is sometimes called the *present value interest factor* for annuities and abbreviated PVIFA(r, t).

The expression for the annuity present value may look a little complicated, but it isn't difficult to use. Notice that the term in square brackets on the second line, $1/(1 + r)^t$, is the same present value factor we've been calculating. In our example from the beginning of this section, the interest rate is 10 percent and there are three years involved. The usual present value factor is thus:

$$Present\ value\ factor = 1/1.1^3 = 1/1.331 = .751315$$

To calculate the annuity present value factor, we just plug this in:

$$Annuity\ present\ value\ factor = (1 - Present\ value\ factor)/r$$
$$= (1 - .751315)/.10$$
$$= .248685/.10 = 2.48685$$

Just as we calculated before, the present value of our $500 annuity is then:

$$Annuity\ present\ value = \$500 \times 2.48685 = \$1,243.43$$

How Much Can You Afford?	**EXAMPLE 6.5**

After carefully going over your budget, you have determined you can afford to pay $632 per month toward a new sports car. You call up your local bank and find out that the going rate is 1 percent per month for 48 months. How much can you borrow?

To determine how much you can borrow, we need to calculate the present value of $632 per month for 48 months at 1 percent per month. The loan payments are in ordinary annuity form, so the annuity present value factor is:

$$Annuity\ PV\ factor = (1 - Present\ value\ factor)/r$$
$$= [1 - (1/1.01^{48})]/.01$$
$$= (1 - .6203)/.01 = 37.9740$$

(continued)

With this factor, we can calculate the present value of the 48 payments of $632 each as:

Present value = $632 × 37.9740 = $24,000

Therefore, $24,000 is what you can afford to borrow and repay.

Annuity Tables   Just as there are tables for ordinary present value factors, there are tables for annuity factors as well. Table 6.1 contains a few such factors; Table A.3 in the appendix to the book contains a larger set. To find the annuity present value factor we calculated just before Example 6.5, look for the row corresponding to three periods and then find the column for 10 percent. The number you see at that intersection should be 2.4869 (rounded to four decimal places), as we calculated. Once again, try calculating a few of these factors yourself and compare your answers to the ones in the table to make sure you know how to do it. If you are using a financial calculator, just enter $1 as the payment and calculate the present value; the result should be the annuity present value factor.

TABLE 6.1

Annuity Present Value
Interest Factors

Number of Periods	Interest Rate			
	5%	10%	15%	20%
1	.9524	.9091	.8696	.8333
2	1.8594	1.7355	1.6257	1.5278
3	2.7232	2.4869	2.2832	2.1065
4	3.5460	3.1699	2.8550	2.5887
5	4.3295	3.7908	3.3522	2.9906

CALCULATOR HINTS

Annuity Present Values

To find annuity present values with a financial calculator, we need to use the [PMT] key (you were probably wondering what it was for). Compared to finding the present value of a single amount, there are two important differences. First, we enter the annuity cash flow using the [PMT] key. Second, we don't enter anything for the future value, [FV]. So, for example, the problem we have been examining is a three-year, $500 annuity. If the discount rate is 10 percent, we need to do the following (after clearing out the calculator!):

Enter	3	10	500		
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for				−1,243.43	

As usual, we get a negative sign on the PV.

SPREADSHEET STRATEGIES

Annuity Present Values

Using a spreadsheet to find annuity present values goes like this:

	A	B	C	D	E	F	G
1							
2	**Using a spreadsheet to find annuity present values**						
3							
4	What is the present value of $500 per year for 3 years if the discount rate is 10 percent?						
5	We need to solve for the unknown present value, so we use the formula PV(rate, nper, pmt, fv).						
6							
7	Payment amount per period:	$500					
8	Number of payments:	3					
9	Discount rate:	0.1					
10							
11	Annuity present value:	**$1,243.43**					
12							
13	The formula entered in cell B11 is =PV(B9,B8,-B7,0); notice that fv is zero and that						
14	pmt has a negative sign on it. Also notice that rate is entered as a decimal, not a percentage.						
15							
16							
17							

Finding the Payment   Suppose you wish to start up a new business that specializes in the latest of health food trends, frozen yak milk. To produce and market your product, the Yakkee Doodle Dandy, you need to borrow $100,000. Because it strikes you as unlikely that this particular fad will be long-lived, you propose to pay off the loan quickly by making five equal annual payments. If the interest rate is 18 percent, what will the payment be?

In this case, we know the present value is $100,000. The interest rate is 18 percent, and there are five years. The payments are all equal, so we need to find the relevant annuity factor and solve for the unknown cash flow:

$$\text{Annuity present value} = \$100,000 = C \times [(1 - \text{Present value factor})/r]$$
$$= C \times \{[1 - (1/1.18^5)]/.18\}$$
$$= C \times [(1 - .4371)/.18]$$
$$= C \times 3.1272$$
$$C = \$100,000/3.1272 = \$31,978$$

Therefore, you'll make five payments of just under $32,000 each.

CALCULATOR HINTS

Annuity Payments

Finding annuity payments is easy with a financial calculator. In our yak example, the PV is $100,000, the interest rate is 18 percent, and there are five years. We find the payment as follows:

Enter	5	18		100,000	
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for			−31,978		

Here, we get a negative sign on the payment because the payment is an outflow for us.

SPREADSHEET STRATEGIES

Annuity Payments

Using a spreadsheet to work the same problem goes like this:

	A	B	C	D	E	F	G
1							
2	**Using a spreadsheet to find annuity payments**						
3							
4	What is the annuity payment if the present value is $100,000, the interest rate is 18 percent, and						
5	there are 5 periods? We need to solve for the unknown payment in an annuity, so we use the						
6	formula PMT(rate, nper, pv, fv).						
7							
8	Annuity present value:	$100,000					
9	Number of payments:	5					
10	Discount rate:	0.18					
11							
12	Annuity payment:	**$31,977.78**					
13							
14	The formula entered in cell B12 is =PMT(B10, B9, -B8,0); notice that fv is zero and that the payment						
15	has a negative sign because it is an outflow to us.						
16							

EXAMPLE 6.6	Finding the Number of Payments

You ran a little short on your spring break vacation, so you put $1,000 on your credit card. You can afford only the minimum payment of $20 per month. The interest rate on the credit card is 1.5 percent per month. How long will you need to pay off the $1,000?

What we have here is an annuity of $20 per month at 1.5 percent per month for some unknown length of time. The present value is $1,000 (the amount you owe today). We need to do a little algebra (or use a financial calculator):

$$\$1,000 = \$20 \times [(1 - \text{Present value factor})/.015]$$
$$(\$1,000/20) \times .015 = 1 - \text{Present value factor}$$
$$\text{Present value factor} = .25 = 1/(1 + r)^t$$
$$1.015^t = 1/.25 = 4$$

At this point, the problem boils down to asking. How long does it take for your money to quadruple at 1.5 percent per month? Based on our previous chapter, the answer is about 93 months:

$$1.015^{93} = 3.99 \approx 4$$

It will take you about $93/12 = 7.75$ years to pay off the $1,000 at this rate. If you use a financial calculator for problems like this, you should be aware that some automatically round up to the next whole period.

CALCULATOR HINTS

Finding the Number of Payments

To solve this one on a financial calculator, do the following:

Enter		1.5	−20	1,000	
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for	93.11				

Notice that we put a negative sign on the payment you must make, and we have solved for the number of months. You still have to divide by 12 to get our answer. Also, some financial calculators won't report a fractional value for N; they automatically (without telling you) round up to the next whole period (not to the nearest value). With a spreadsheet, use the function =NPER(rate,pmt,pv,fv); be sure to put in a zero for fv and to enter −20 as the payment.

Finding the Rate   The last question we might want to ask concerns the interest rate implicit in an annuity. For example, an insurance company offers to pay you $1,000 per year for 10 years if you will pay $6,710 up front. What rate is implicit in this 10-year annuity?

In this case, we know the present value ($6,710), we know the cash flows ($1,000 per year), and we know the life of the investment (10 years). What we don't know is the discount rate:

$$\$6{,}710 = \$1{,}000 \times [(1 - \text{Present value factor})/r]$$
$$\$6{,}710/1{,}000 = 6.71 = \{1 - [1/(1 + r)^{10}]\}/r$$

So, the annuity factor for 10 periods is equal to 6.71, and we need to solve this equation for the unknown value of r. Unfortunately, this is mathematically impossible to do directly. The only way to do it is to use a table or trial and error to find a value for r.

If you look across the row corresponding to 10 periods in Table A.3, you will see a factor of 6.7101 for 8 percent, so we see right away that the insurance company is offering just about 8 percent. Alternatively, we could just start trying different values until we got very close to the answer. Using this trial-and-error approach can be a little tedious, but fortunately machines are good at that sort of thing.[1]

To illustrate how to find the answer by trial and error, suppose a relative of yours wants to borrow $3,000. She offers to repay you $1,000 every year for four years. What interest rate are you being offered?

The cash flows here have the form of a four-year, $1,000 annuity. The present value is $3,000. We need to find the discount rate, r. Our goal in doing so is primarily to give you a feel for the relationship between annuity values and discount rates.

We need to start somewhere, and 10 percent is probably as good a place as any to begin. At 10 percent, the annuity factor is:

Annuity present value factor $= [1 - (1/1.10^4)]/.10 = 3.1699$

----

[1]Financial calculators rely on trial and error to find the answer. That's why they sometimes appear to be "thinking" before coming up with the answer. Actually, it is possible to directly solve for r if there are fewer than five periods, but it's usually not worth the trouble.

The present value of the cash flows at 10 percent is thus:

Present value = $1,000 × 3.1699 = $3,169.90

You can see that we're already in the right ballpark.

Is 10 percent too high or too low? Recall that present values and discount rates move in opposite directions: Increasing the discount rate lowers the PV and vice versa. Our present value here is too high, so the discount rate is too low. If we try 12 percent, we're almost there:

Present value = $1,000 × {[1 − (1/1.12^4)]/.12} = $3,037.35

We are still a little low on the discount rate (because the PV is a little high), so we'll try 13 percent:

Present value = $1,000 × {[1 − (1/1.13^4)]/.13} = $2,974.47

This is less than $3,000, so we now know that the answer is between 12 percent and 13 percent, and it looks to be about 12.5 percent. For practice, work at it for a while longer and see if you find that the answer is about 12.59 percent.

To illustrate a situation in which finding the unknown rate can be useful, let us consider that the Tri-State Megabucks lottery in Maine, Vermont, and New Hampshire offers you a choice of how to take your winnings (most lotteries do this). In a recent drawing, participants were offered the option of receiving a lump sum payment of $250,000 or an annuity of $500,000 to be received in equal installments over a 25-year period. (At the time, the lump sum payment was always half the annuity option.) Which option was better?

To answer, suppose you were to compare $250,000 today to an annuity of $500,000/25 = $20,000 per year for 25 years. At what rate do these have the same value? This is the same problem we've been looking at; we need to find the unknown rate, r, for a present value of $250,000, a $20,000 payment, and a 25-year period. If you grind through the calculations (or get a little machine assistance), you should find that the unknown rate is about 6.24 percent. You should take the annuity option if that rate is attractive relative to other investments available to you. Notice that we have ignored taxes in this example, and taxes can significantly affect our conclusion. Be sure to consult your tax adviser anytime you win the lottery.

CALCULATOR HINTS

Finding the Rate

Alternatively, you could use a financial calculator to do the following:

Enter	4		1,000	−3,000	
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for		12.59			

Notice that we put a negative sign on the present value (why?). With a spreadsheet, use the function =RATE(nper,pmt,pv,fv); be sure to put in a zero for fv and to enter 1,000 as the payment and −3,000 as the pv.

FUTURE VALUE FOR ANNUITIES

On occasion, it's also handy to know a shortcut for calculating the future value of an annuity. As you might guess, there are future value factors for annuities as well as present value factors. In general, here is the future value factor for an annuity:

$$\text{Annuity FV factor} = (\text{Future value factor} - 1)/r$$
$$= [(1 + r)^t - 1]/r \qquad \text{[6.2]}$$

To see how we use annuity future value factors, suppose you plan to contribute $2,000 every year to a retirement account paying 8 percent. If you retire in 30 years, how much will you have?

The number of years here, t, is 30, and the interest rate, r, is 8 percent; so we can calculate the annuity future value factor as:

$$\text{Annuity FV factor} = (\text{Future value factor} - 1)/r$$
$$= (1.08^{30} - 1)/.08$$
$$= (10.0627 - 1)/.08$$
$$= 113.2832$$

The future value of this 30-year, $2,000 annuity is thus:

$$\text{Annuity future value} = \$2,000 \times 113.28$$
$$= \$226,566$$

Sometimes we need to find the unknown rate, r, in the context of an annuity future value. For example, if you had invested $100 per month in stocks over the 25-year period ended December 1978, your investment would have grown to $76,374. This period had the *worst* stretch of stock returns of any 25-year period between 1925 and 2005. How bad was it?

CALCULATOR HINTS

Future Values of Annuities

Of course, you could solve this problem using a financial calculator by doing the following:

Enter	30	8	−2,000		
	N	**I/Y**	**PMT**	**PV**	**FV**
Solve for					226,566

Notice that we put a negative sign on the payment (why?). With a spreadsheet, use the function =FV(rate,nper, pmt,pv); be sure to put in a zero for pv and to enter −2,000 as the payment.

Here we have the cash flows ($100 per month), the *future* value ($76,374), and the time period (25 years, or 300 months). We need to find the implicit rate, r:

$$\$76,374 = \$100 \times [(\text{Future value factor} - 1)/r]$$
$$763.74 = [(1 + r)^{300} - 1]/r$$

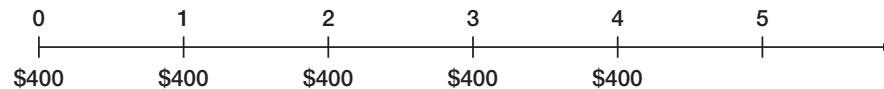Because this is the worst period, let's try 1 percent:

$$\text{Annuity future value factor} = (1.01^{300} - 1)/.01 = 1,878.85$$

We see that 1 percent is too high. From here, it's trial and error. See if you agree that r is about .55 percent per month. As you will see later in the chapter, this works out to be about 6.8 percent per year.

A NOTE ABOUT ANNUITIES DUE

annuity due
An annuity for which the cash flows occur at the beginning of the period.

So far we have only discussed ordinary annuities. These are the most important, but there is a fairly common variation. Remember that with an ordinary annuity, the cash flows occur at the end of each period. When you take out a loan with monthly payments, for example, the first loan payment normally occurs one month after you get the loan. However, when you lease an apartment, the first lease payment is usually due immediately. The second payment is due at the beginning of the second month, and so on. A lease is an example of an **annuity due**. An annuity due is an annuity for which the cash flows occur at the beginning of each period. Almost any type of arrangement in which we have to prepay the same amount each period is an annuity due.

There are several different ways to calculate the value of an annuity due. With a financial calculator, you simply switch it into "due" or "beginning" mode. Remember to switch it back when you are done! Another way to calculate the present value of an annuity due can be illustrated with a time line. Suppose an annuity due has five payments of $400 each, and the relevant discount rate is 10 percent. The time line looks like this:

0	1	2	3	4	5
$400	$400	$400	$400	$400	

Time value applications abound on the Web. See, for example, www.collegeboard.com, www.1stmortgagedirectory.com, and personal.fidelity.com.

Notice how the cash flows here are the same as those for a *four*-year ordinary annuity, except that there is an extra $400 at Time 0. For practice, check to see that the value of a four-year ordinary annuity at 10 percent is $1,267.95. If we add on the extra $400, we get $1,667.95, which is the present value of this annuity due.

There is an even easier way to calculate the present or future value of an annuity due. If we assume cash flows occur at the end of each period when they really occur at the beginning, then we discount each one by one period too many. We could fix this by simply multiplying our answer by $(1 + r)$, where r is the discount rate. In fact, the relationship between the value of an annuity due and an ordinary annuity is just this:

$$\text{Annuity due value} = \text{Ordinary annuity value} \times (1 + r) \qquad \text{[6.3]}$$

This works for both present and future values, so calculating the value of an annuity due involves two steps: (1) Calculate the present or future value as though it were an ordinary annuity, and (2) multiply your answer by $(1 + r)$.

PERPETUITIES

perpetuity
An annuity in which the cash flows continue forever.

consol
A type of perpetuity.

We've seen that a series of level cash flows can be valued by treating those cash flows as an annuity. An important special case of an annuity arises when the level stream of cash flows continues forever. Such an asset is called a **perpetuity** because the cash flows are perpetual. Perpetuities are also called **consols**, particularly in Canada and the United Kingdom. See Example 6.7 for an important example of a perpetuity.

Because a perpetuity has an infinite number of cash flows, we obviously can't compute its value by discounting each one. Fortunately, valuing a perpetuity turns out to be the easiest possible case. The present value of a perpetuity is simply:

$$\text{PV for a perpetuity} = C/r \qquad \text{[6.4]}$$

I.   Symbols:
PV = Present value, what future cash flows are worth today
FV_t = Future value, what cash flows are worth in the future
r = Interest rate, rate of return, or discount rate per period—typically, but not always, one year
t = Number of periods—typically, but not always, the number of years
C = Cash amount
II.   Future Value of C per Period for t Periods at r Percent per Period:
$FV_t = C \times \{[(1 + r)^t - 1]/r\}$
A series of identical cash flows is called an *annuity*, and the term $[(1 + r)^t - 1]/r$ is called the *annuity future value factor*.
III.   Present Value of C per Period for t Periods at r Percent per Period:
$PV = C \times \{1 - [1/(1 + r)^t]\}/r$
The term $\{1 - [1/(1 + r)^t]\}/r$ is called the *annuity present value factor*.
IV.   Present Value of a Perpetuity of C per Period:
$PV = C/r$
A *perpetuity* has the same cash flow every year forever.

TABLE 6.2

Summary of Annuity and Perpetuity Calculations

For example, an investment offers a perpetual cash flow of $500 every year. The return you require on such an investment is 8 percent. What is the value of this investment? The value of this perpetuity is:

Perpetuity $PV = C/r = \$500/.08 = \$6,250$

For future reference, Table 6.2 contains a summary of the annuity and perpetuity basic calculations we described. By now, you probably think that you'll just use online calculators to handle annuity problems. Before you do, see our nearby *Work the Web* box!

Preferred Stock	**EXAMPLE 6.7**

Preferred stock (or preference stock) is an important example of a perpetuity. When a corporation sells preferred stock, the buyer is promised a fixed cash dividend every period (usually every quarter) forever. This dividend must be paid before any dividend can be paid to regular stockholders—hence the term *preferred.*

Suppose the Fellini Co. wants to sell preferred stock at $100 per share. A similar issue of preferred stock already outstanding has a price of $40 per share and offers a dividend of $1 every quarter. What dividend will Fellini have to offer if the preferred stock is going to sell?

The issue that is already out has a present value of $40 and a cash flow of $1 every quarter forever. Because this is a perpetuity:

Present value = $40 = $1 × (1/r)

r = 2.5%

To be competitive, the new Fellini issue will also have to offer 2.5 percent *per quarter;* so if the present value is to be $100, the dividend must be such that:

Present value = $100 = C × (1/.025)

C = $2.50 (per quarter)

GROWING ANNUITIES AND PERPETUITIES

Annuities commonly have payments that grow over time. Suppose, for example, that we are looking at a lottery payout over a 20-year period. The first payment, made one year from now, will be $200,000. Every year thereafter, the payment will grow by 5 percent, so the payment in the second year will be $200,000 × 1.05 = $210,000. The payment in the third year will be $210,000 × 1.05 = $220,500, and so on. What's the present value if the appropriate discount rate is 11 percent?

If we use the symbol g to represent the growth rate, we can calculate the value of a growing annuity using a modified version of our regular annuity formula:

$$\text{Growing annuity present value} = C \times \left[\frac{1 - \left(\frac{1 + g}{1 + r} \right)^t}{r - g} \right] \qquad [6.5]$$

Plugging in the numbers from our lottery example (and letting $g = .05$), we get:

$$PV = \$200,000 \times \left[\frac{1 - \left(\frac{1 + .05}{1 + .11} \right)^{20}}{.11 - .05} \right] = \$200,000 \times 11.18169 = \$2,236,337.06$$

WORK THE WEB

As we discussed in the previous chapter, many Web sites have financial calculators. One of these sites is MoneyChimp, which is located at www.moneychimp.com. Suppose you are lucky enough to have $2,000,000. You think you will be able to earn a 9 percent return. How much can you withdraw each year for the next 30 years? Here is what MoneyChimp says:

Annuity Calculator

Inputs		
Starting Principal:	$	2,000,000.00
Growth Rate:		9 %
Years to Pay Out:		30

Calculate

Results		
Annual Payout Amount:	$	178,598.81

According to the MoneyChimp calculator, the answer is $178,598.81. How important is it to understand what you are doing? Calculate this one for yourself, and you should get $194,672.70. Which one is right? You are, of course! What's going on is that MoneyChimp assumes (but does not tell you) that the annuity is in the form of an annuity due, not an ordinary annuity. Recall that with an annuity due, the payments occur at the beginning of the period rather than the end of the period. The moral of the story is clear: *caveat calculator.*

There is also a formula for the present value of a growing perpetuity:

$$\text{Growing perpetuity present value} = C \times \left[\frac{1}{r - g}\right] = \frac{C}{r - g} \qquad \text{[6.6]}$$

In our lottery example, now suppose the payments continue forever. In this case, the present value is:

$$\text{PV} = \$200{,}000 \times \frac{1}{.11 - .05} = \$200{,}000 \times 16.6667 = \$3{,}333{,}333.33$$

The notion of a growing perpetuity may seem a little odd because the payments get bigger every period forever; but, as we will see in a later chapter, growing perpetuities play a key role in our analysis of stock prices.

Before we go on, there is one important note about our formulas for growing annuities and perpetuities. In both cases, the cash flow in the formula, C, is the cash flow that is going to occur exactly one period from today.

Concept Questions

6.2a In general, what is the present value of an annuity of C dollars per period at a discount rate of r per period? The future value?

6.2b In general, what is the present value of a perpetuity?

Comparing Rates: The Effect of Compounding   6.3

The next issue we need to discuss has to do with the way interest rates are quoted. This subject causes a fair amount of confusion because rates are quoted in many different ways. Sometimes the way a rate is quoted is the result of tradition, and sometimes it's the result of legislation. Unfortunately, at times, rates are quoted in deliberately deceptive ways to mislead borrowers and investors. We will discuss these topics in this section.

EFFECTIVE ANNUAL RATES AND COMPOUNDING

If a rate is quoted as 10 percent compounded semiannually, this means the investment actually pays 5 percent every six months. A natural question then arises: Is 5 percent every six months the same thing as 10 percent per year? It's easy to see that it is not. If you invest $1 at 10 percent per year, you will have $1.10 at the end of the year. If you invest at 5 percent every six months, then you'll have the future value of $1 at 5 percent for two periods:

$$\$1 \times 1.05^2 = \$1.1025$$

This is $.0025 more. The reason is simple: Your account was credited with $1 \times .05 = 5 cents in interest after six months. In the following six months, you earned 5 percent on that nickel, for an extra $5 \times .05 = .25$ cents.

As our example illustrates, 10 percent compounded semiannually is actually equivalent to 10.25 percent per year. Put another way, we would be indifferent between

stated interest rate
The interest rate expressed in terms of the interest payment made each period. Also known as the *quoted interest rate*.

effective annual rate (EAR)
The interest rate expressed as if it were compounded once per year.

10 percent compounded semiannually and 10.25 percent compounded annually. Anytime we have compounding during the year, we need to be concerned about what the rate really is.

In our example, the 10 percent is called a **stated**, or **quoted**, **interest rate**. Other names are used as well. The 10.25 percent, which is actually the rate you will earn, is called the **effective annual rate (EAR)**. To compare different investments or interest rates, we will always need to convert to effective rates. Some general procedures for doing this are discussed next.

CALCULATING AND COMPARING EFFECTIVE ANNUAL RATES

To see why it is important to work only with effective rates, suppose you've shopped around and come up with the following three rates:

> Bank A: 15 percent compounded daily
> Bank B: 15.5 percent compounded quarterly
> Bank C: 16 percent compounded annually

Which of these is the best if you are thinking of opening a savings account? Which of these is best if they represent loan rates?

To begin, Bank C is offering 16 percent per year. Because there is no compounding during the year, this is the effective rate. Bank B is actually paying $.155/4 = .03875$ or 3.875 percent per quarter. At this rate, an investment of \$1 for four quarters would grow to:

$$\$1 \times 1.03875^4 = \$1.1642$$

The EAR, therefore, is 16.42 percent. For a saver, this is much better than the 16 percent rate Bank C is offering; for a borrower, it's worse.

Bank A is compounding every day. This may seem a little extreme, but it is common to calculate interest daily. In this case, the daily interest rate is actually:

$$.15/365 = .000411$$

This is .0411 percent per day. At this rate, an investment of \$1 for 365 periods would grow to:

$$\$1 \times 1.000411^{365} = \$1.1618$$

The EAR is 16.18 percent. This is not as good as Bank B's 16.42 percent for a saver, and not as good as Bank C's 16 percent for a borrower.

This example illustrates two things. First, the highest quoted rate is not necessarily the best. Second, compounding during the year can lead to a significant difference between the quoted rate and the effective rate. Remember that the effective rate is what you get or what you pay.

If you look at our examples, you see that we computed the EARs in three steps. We first divided the quoted rate by the number of times that the interest is compounded. We then added 1 to the result and raised it to the power of the number of times the interest is compounded. Finally, we subtracted the 1. If we let m be the number of times the interest is compounded during the year, these steps can be summarized simply as:

$$EAR = [1 + (\text{Quoted rate}/m)]^m - 1 \qquad [6.7]$$

For example, suppose you are offered 12 percent compounded monthly. In this case, the interest is compounded 12 times a year; so m is 12. You can calculate the effective rate as:

$$\text{EAR} = [1 + (\text{Quoted rate}/m)]^m - 1$$
$$= [1 + (.12/12)]^{12} - 1$$
$$= 1.01^{12} - 1$$
$$= 1.126825 - 1$$
$$= 12.6825\%$$

What's the EAR?	**EXAMPLE 6.8**

A bank is offering 12 percent compounded quarterly. If you put $100 in an account, how much will you have at the end of one year? What's the EAR? How much will you have at the end of two years?

The bank is effectively offering $12\%/4 = 3\%$ every quarter. If you invest $100 for four periods at 3 percent per period, the future value is:

$$\text{Future value} = \$100 \times 1.03^4$$
$$= \$100 \times 1.1255$$
$$= \$112.55$$

The EAR is 12.55 percent: $100 \times (1 + .1255) = \$112.55$.

We can determine what you would have at the end of two years in two different ways. One way is to recognize that two years is the same as eight quarters. At 3 percent per quarter, after eight quarters, you would have:

$$\$100 \times 1.03^8 = \$100 \times 1.2668 = \$126.68$$

Alternatively, we could determine the value after two years by using an EAR of 12.55 percent; so after two years you would have:

$$\$100 \times 1.1255^2 = \$100 \times 1.2688 = \$126.68$$

Thus, the two calculations produce the same answer. This illustrates an important point. Anytime we do a present or future value calculation, the rate we use must be an actual or effective rate. In this case, the actual rate is 3 percent per quarter. The effective annual rate is 12.55 percent. It doesn't matter which one we use once we know the EAR.

Quoting a Rate	**EXAMPLE 6.9**

Now that you know how to convert a quoted rate to an EAR, consider going the other way. As a lender, you know you want to actually earn 18 percent on a particular loan. You want to quote a rate that features monthly compounding. What rate do you quote?

In this case, we know the EAR is 18 percent, and we know this is the result of monthly compounding. Let q stand for the quoted rate. We thus have:

$$\text{EAR} = [1 + (\text{Quoted rate}/m)]^m - 1$$
$$.18 = [1 + (q/12)]^{12} - 1$$
$$1.18 = [1 + (q/12)]^{12}$$

(continued)

We need to solve this equation for the quoted rate. This calculation is the same as the ones we did to find an unknown interest rate in Chapter 5:

$$1.18^{(1/12)} = 1 + (q/12)$$
$$1.18^{.08333} = 1 + (q/12)$$
$$1.0139 = 1 + (q/12)$$
$$q = .0139 \times 12$$
$$= 16.68\%$$

Therefore, the rate you would quote is 16.68 percent, compounded monthly.

EARs AND APRs

annual percentage rate (APR)
The interest rate charged per period multiplied by the number of periods per year.

Sometimes it's not altogether clear whether a rate is an effective annual rate. A case in point concerns what is called the **annual percentage rate (APR)** on a loan. Truth-in-lending laws in the United States require that lenders disclose an APR on virtually all consumer loans. This rate must be displayed on a loan document in a prominent and unambiguous way.

Given that an APR must be calculated and displayed, an obvious question arises: Is an APR an effective annual rate? Put another way, if a bank quotes a car loan at 12 percent APR, is the consumer actually paying 12 percent interest? Surprisingly, the answer is no. There is some confusion over this point, which we discuss next.

The confusion over APRs arises because lenders are required by law to compute the APR in a particular way. By law, the APR is simply equal to the interest rate per period multiplied by the number of periods in a year. For example, if a bank is charging 1.2 percent per month on car loans, then the APR that must be reported is $1.2\% \times 12 = 14.4\%$. So, an APR is in fact a quoted, or stated, rate in the sense we've been discussing. For example, an APR of 12 percent on a loan calling for monthly payments is really 1 percent per month. The EAR on such a loan is thus:

$$EAR = [1 + (APR/12)]^{12} - 1$$
$$= 1.01^{12} - 1 = 12.6825\%$$

EXAMPLE 6.10	**What Rate Are You Paying?**

Depending on the issuer, a typical credit card agreement quotes an interest rate of 18 percent APR. Monthly payments are required. What is the actual interest rate you pay on such a credit card?

Based on our discussion, an APR of 18 percent with monthly payments is really .18/12 = .015 or 1.5 percent per month. The EAR is thus:

$$EAR = [1 + (.18/12)]^{12} - 1$$
$$= 1.015^{12} - 1$$
$$= 1.1956 - 1$$
$$= 19.56\%$$

This is the rate you actually pay.

It is somewhat ironic that truth-in-lending laws sometimes require lenders to be *un*truthful about the actual rate on a loan. There are also truth-in-saving laws that require banks and other borrowers to quote an "annual percentage yield," or APY, on things like savings accounts.

To make things a little confusing, an APY is an EAR. As a result, by law, the rates quoted to borrowers (APRs) and those quoted to savers (APYs) are not computed the same way.

There can be a huge difference between the APR and EAR when interest rate are large. For example, consider "payday loans." Payday loans are short-term loans made to consumers, often for less than two weeks, and are offered by companies such as AmeriCash Advance and National Payday. The loans work like this: You write a check today that is postdated (the date on the check is in the future) and give it to the company. They give you some cash. When the check date arrives, you either to the store and pay the cash amount of the check, or the company cashes it (or else automatically renews the loan).

For example, AmeriCash Advance allows you to write a check for $125 dated 15 days in the future, for which they give you $100 today. So what are the APR and EAR of this arrangement? First, we need to find the interest rate, which we can find by the FV equation as follows:

$$FV = PV \times (1 + r)^1$$
$$\$125 = \$100 \times (1 + r)^1$$
$$1.25 = (1 + r)$$
$$r = .25 \text{ or } 25\%$$

That doesn't seem too bad until you remember this is the interest rate for *15 days!* The APR of the loan is:

$$APR = .25 \times 365/15$$
$$APR = 6.08333 \text{ or } 608.33\%$$

And the EAR for this loan is:

$$EAR = (1 + \text{Quoted rate}/m)^m - 1$$
$$EAR = (1 + .25)^{365/15} - 1$$
$$EAR = 227.1096 \text{ or } 22,710.96\%$$

Now that's an interest rate! Just to see what a difference a day (or three) makes, let's look at National Payday's terms. This company will allow you to write a postdated check for the same amount, but will give you 18 days to repay. Check for yourself that the APR of this arrangement is 506.94 percent and the EAR is 9,128.26 percent. Still not a loan we would like to take out!

TAKING IT TO THE LIMIT: A NOTE ABOUT CONTINUOUS COMPOUNDING

If you made a deposit in a savings account, how often could your money be compounded during the year? If you think about it, there isn't really any upper limit. We've seen that daily compounding, for example, isn't a problem. There is no reason to stop here, however. We could compound every hour or minute or second. How high would the EAR get in this case? Table 6.3 illustrates the EARs that result as 10 percent is compounded at shorter and shorter intervals. Notice that the EARs do keep getting larger, but the differences get very small.

As the numbers in Table 6.3 seem to suggest, there is an upper limit to the EAR. If we let q stand for the quoted rate, then, as the number of times the interest is compounded gets extremely large, the EAR approaches:

$$EAR = e^q - 1 \hspace{4cm} \text{[6.8]}$$

TABLE 6.3

Compounding Frequency and Effective Annual Rates

Compounding Period	Number of Times Compounded	Effective Annual Rate
Year	1	10.00000%
Quarter	4	10.38129
Month	12	10.47131
Week	52	10.50648
Day	365	10.51558
Hour	8,760	10.51703
Minute	525,600	10.51709

where e is the number 2.71828 (look for a key labeled "e^x" on your calculator). For example, with our 10 percent rate, the highest possible EAR is:

$$EAR = e^q - 1$$
$$= 2.71828^{.10} - 1$$
$$= 1.1051709 - 1$$
$$= 10.51709\%$$

In this case, we say that the money is continuously, or instantaneously, compounded. Interest is being credited the instant it is earned, so the amount of interest grows continuously.

EXAMPLE 6.11

What's the Law?

At one time, commercial banks and savings and loan associations (S&Ls) were restricted in the interest rates they could offer on savings accounts. Under what was known as Regulation Q, S&Ls were allowed to pay at most 5.5 percent, and banks were not allowed to pay more than 5.25 percent (the idea was to give the S&Ls a competitive advantage; it didn't work). The law did not say how often these rates could be compounded, however. Under Regulation Q, then, what were the maximum allowed interest rates?

   The maximum allowed rates occurred with continuous, or instantaneous, compounding. For the commercial banks, 5.25 percent compounded continuously would be:

$$EAR = e^{.0525} - 1$$
$$= 2.71828^{.0525} - 1$$
$$= 1.0539026 - 1$$
$$= 5.39026\%$$

This is what banks could actually pay. Check for yourself to see that S&Ls could effectively pay 5.65406 percent.

Concept Questions

6.3a If an interest rate is given as 12 percent compounded daily, what do we call this rate?

6.3b What is an APR? What is an EAR? Are they the same thing?

6.3c In general, what is the relationship between a stated interest rate and an effective interest rate? Which is more relevant for financial decisions?

6.3d What does continuous compounding mean?

Loan Types and Loan Amortization   **6.4**

Whenever a lender extends a loan, some provision will be made for repayment of the principal (the original loan amount). A loan might be repaid in equal installments, for example, or it might be repaid in a single lump sum. Because the way that the principal and interest are paid is up to the parties involved, there are actually an unlimited number of possibilities.

In this section, we describe a few forms of repayment that come up quite often, and more complicated forms can usually be built up from these. The three basic types of loans are pure discount loans, interest-only loans, and amortized loans. Working with these loans is a very straightforward application of the present value principles that we have already developed.

PURE DISCOUNT LOANS

The *pure discount loan* is the simplest form of loan. With such a loan, the borrower receives money today and repays a single lump sum at some time in the future. A one-year, 10 percent pure discount loan, for example, would require the borrower to repay $1.10 in one year for every dollar borrowed today.

Because a pure discount loan is so simple, we already know how to value one. Suppose a borrower was able to repay $25,000 in five years. If we, acting as the lender, wanted a 12 percent interest rate on the loan, how much would we be willing to lend? Put another way, what value would we assign today to that $25,000 to be repaid in five years? Based on our work in Chapter 5, we know the answer is just the present value of $25,000 at 12 percent for five years:

$$\text{Present value} = \$25{,}000/1.12^5$$
$$= \$25{,}000/1.7623$$
$$= \$14{,}186$$

Pure discount loans are common when the loan term is short—say a year or less. In recent years, they have become increasingly common for much longer periods.

Treasury Bills	**EXAMPLE 6.12**

When the U.S. government borrows money on a short-term basis (a year or less), it does so by selling what are called *Treasury bills,* or *T-bills* for short. A T-bill is a promise by the government to repay a fixed amount at some time in the future—for example, 3 months or 12 months.

Treasury bills are pure discount loans. If a T-bill promises to repay $10,000 in 12 months, and the market interest rate is 7 percent, how much will the bill sell for in the market?

Because the going rate is 7 percent, the T-bill will sell for the present value of $10,000 to be repaid in one year at 7 percent:

$$\text{Present value} = \$10{,}000/1.07 = \$9{,}345.79$$

INTEREST-ONLY LOANS

A second type of loan repayment plan calls for the borrower to pay interest each period and to repay the entire principal (the original loan amount) at some point in the future. Loans

with such a repayment plan are called *interest-only loans.* Notice that if there is just one period, a pure discount loan and an interest-only loan are the same thing.

For example, with a three-year, 10 percent, interest-only loan of $1,000, the borrower would pay $1,000 × .10 = $100 in interest at the end of the first and second years. At the end of the third year, the borrower would return the $1,000 along with another $100 in interest for that year. Similarly, a 50-year interest-only loan would call for the borrower to pay interest every year for the next 50 years and then repay the principal. In the extreme, the borrower pays the interest every period forever and never repays any principal. As we discussed earlier in the chapter, the result is a perpetuity.

Most corporate bonds have the general form of an interest-only loan. Because we will be considering bonds in some detail in the next chapter, we will defer further discussion of them for now.

AMORTIZED LOANS

With a pure discount or interest-only loan, the principal is repaid all at once. An alternative is an *amortized loan,* with which the lender may require the borrower to repay parts of the loan amount over time. The process of providing for a loan to be paid off by making regular principal reductions is called *amortizing* the loan.

A simple way of amortizing a loan is to have the borrower pay the interest each period plus some fixed amount. This approach is common with medium-term business loans. For example, suppose a business takes out a $5,000, five-year loan at 9 percent. The loan agreement calls for the borrower to pay the interest on the loan balance each year and to reduce the loan balance each year by $1,000. Because the loan amount declines by $1,000 each year, it is fully paid in five years.

In the case we are considering, notice that the total payment will decline each year. The reason is that the loan balance goes down, resulting in a lower interest charge each year, whereas the $1,000 principal reduction is constant. For example, the interest in the first year will be $5,000 × .09 = $450. The total payment will be $1,000 + 450 = $1,450. In the second year, the loan balance is $4,000, so the interest is $4,000 × .09 = $360, and the total payment is $1,360. We can calculate the total payment in each of the remaining years by preparing a simple *amortization schedule* as follows:

Year	Beginning Balance	Total Payment	Interest Paid	Principal Paid	Ending Balance
1	$5,000	$1,450	$ 450	$1,000	$4,000
2	4,000	1,360	360	1,000	3,000
3	3,000	1,270	270	1,000	2,000
4	2,000	1,180	180	1,000	1,000
5	1,000	1,090	90	1,000	0
Totals		$6,350	$1,350	$5,000	

Notice that in each year, the interest paid is given by the beginning balance multiplied by the interest rate. Also notice that the beginning balance is given by the ending balance from the previous year.

Probably the most common way of amortizing a loan is to have the borrower make a single, fixed payment every period. Almost all consumer loans (such as car loans) and mortgages work this way. For example, suppose our five-year, 9 percent, $5,000 loan was amortized this way. How would the amortization schedule look?

We first need to determine the payment. From our discussion earlier in the chapter, we know that this loan's cash flows are in the form of an ordinary annuity. In this case, we can solve for the payment as follows:

$$\$5,000 = C \times \{[1 - (1/1.09^5)]/.09\}$$
$$= C \times [(1 - .6499)/.09]$$

This gives us:

$$C = \$5,000/3.8897$$
$$= \$1,285.46$$

The borrower will therefore make five equal payments of $1,285.46. Will this pay off the loan? We will check by filling in an amortization schedule.

In our previous example, we knew the principal reduction each year. We then calculated the interest owed to get the total payment. In this example, we know the total payment. We will thus calculate the interest and then subtract it from the total payment to calculate the principal portion in each payment.

In the first year, the interest is $450, as we calculated before. Because the total payment is $1,285.46, the principal paid in the first year must be:

$$\text{Principal paid} = \$1,285.46 - 450 = \$835.46$$

The ending loan balance is thus:

$$\text{Ending balance} = \$5,000 - 835.46 = \$4,164.54$$

The interest in the second year is $4,164.54 \times .09 = \$374.81$, and the loan balance declines by $1,285.46 - 374.81 = \$910.65$. We can summarize all of the relevant calculations in the following schedule:

Year	Beginning Balance	Total Payment	Interest Paid	Principal Paid	Ending Balance
1	$5,000.00	$1,285.46	$ 450.00	$ 835.46	$4,164.54
2	4,164.54	1,285.46	374.81	910.65	3,253.88
3	3,253.88	1,285.46	292.85	992.61	2,261.27
4	2,261.27	1,285.46	203.51	1,081.95	1,179.32
5	1,179.32	1,285.46	106.14	1,179.32	0.00
Totals		$6,427.30	$1,427.31	$5,000.00	

Because the loan balance declines to zero, the five equal payments do pay off the loan. Notice that the interest paid declines each period. This isn't surprising because the loan balance is going down. Given that the total payment is fixed, the principal paid must be rising each period.

If you compare the two loan amortizations in this section, you will see that the total interest is greater for the equal total payment case: $1,427.31 versus $1,350. The reason for this is that the loan is repaid more slowly early on, so the interest is somewhat higher. This doesn't mean that one loan is better than the other; it simply means that one is effectively paid off faster than the other. For example, the principal reduction in the first year is $835.46 in the equal total payment case as compared to $1,000 in the first case. Many Web sites offer loan amortization schedules. See our nearby *Work the Web* box for an example.

We will close this chapter with an example that may be of particular relevance. Federal Stafford loans are an important source of financing for many college students, helping to

WORK THE WEB

Preparing an amortization table is one of the more tedious time value of money applications. Using a spreadsheet makes it relatively easy, but there are also Web sites available that will prepare an amortization very quickly and simply. One such site is www.bankrate.com. This site has a mortgage calculator for home loans, but the same calculations apply to most other types of loans such as car loans and student loans. Suppose you graduate with a student loan of $20,000 and will repay the loan over the next 10 years at 6.25 percent. What are your monthly payments? Using the calculator we get:

Monthly mortgage payment and amortization calculator

Calculate your monthly mortgage payment. Click on the "?" next to the input box for an item to get help on that item. Find rates in your area

Mortgage amount $ `20000.00` [?] (Do not use commas.)

Mortgage term `10` [?] years

Interest rate `6.25` [?] % per year

Mortgage start date `Dec ▾` `1 ▾`, `2005 ▾`

Monthly mortgage payment $ `224.56` [?]

[Calculate]

[Show/Recalculate Amortization Table] [?]

Try this example yourself and click the "Show/Recalculate Amortization Table" button. You will find that your first payment will consist of $120.39 in principal and $104.17 in interest. Over the life of the loan you will pay a total of $6,947.22 in interest.

EXAMPLE 6.13 | **Partial Amortization, or "Bite the Bullet"**

A common arrangement in real estate lending might call for a 5-year loan with, say, a 15-year amortization. What this means is that the borrower makes a payment every month of a fixed amount based on a 15-year amortization. However, after 60 months, the borrower makes a single, much larger payment called a "balloon" or "bullet" to pay off the loan. Because the monthly payments don't fully pay off the loan, the loan is said to be partially amortized.

Suppose we have a $100,000 commercial mortgage with a 12 percent APR and a 20-year (240-month) amortization. Further suppose the mortgage has a five-year balloon. What will the monthly payment be? How big will the balloon payment be?

(continued)

The monthly payment can be calculated based on an ordinary annuity with a present value of $100,000. There are 240 payments, and the interest rate is 1 percent per month. The payment is:

$$\$100,000 = C \times [1 - (1/1.01^{240})/.01]$$
$$= C \times 90.8194$$
$$C = \$1,101.09$$

Now, there is an easy way and a hard way to determine the balloon payment. The hard way is to actually amortize the loan for 60 months to see what the balance is at that time. The easy way is to recognize that after 60 months, we have a $240 - 60 = 180$-month loan. The payment is still $1,101.09 per month, and the interest rate is still 1 percent per month. The loan balance is thus the present value of the remaining payments:

$$\text{Loan balance} = \$1,101.09 \times [1 - (1/1.01^{180})/.01]$$
$$= \$1,101.09 \times 83.3217$$
$$= \$91,744.69$$

The balloon payment is a substantial $91,744. Why is it so large? To get an idea, consider the first payment on the mortgage. The interest in the first month is $100,000 \times .01 = \$1,000$. Your payment is $1,101.09, so the loan balance declines by only $101.09. Because the loan balance declines so slowly, the cumulative "pay down" over five years is not great.

cover the cost of tuition, books, new cars, condominiums, and many other things. Sometimes students do not seem to fully realize that Stafford loans have a serious drawback: They must be repaid in monthly installments, usually beginning six months after the student leaves school.

Some Stafford loans are subsidized, meaning that the interest does not begin to accrue until repayment begins (this is a good thing). If you are a dependent undergraduate student under this particular option, the total debt you can run up is, at most, $23,000. The maximum interest rate is 8.25 percent, or $8.25/12 = 0.6875$ percent per month. Under the "standard repayment plan," the loans are amortized over 10 years (subject to a minimum payment of $50).

Suppose you max out borrowing under this program and also get stuck paying the maximum interest rate. Beginning six months after you graduate (or otherwise depart the ivory tower), what will your monthly payment be? How much will you owe after making payments for four years?

Given our earlier discussions, see if you don't agree that your monthly payment assuming a $23,000 total loan is $282.10 per month. Also, as explained in Example 6.13, after making payments for four years, you still owe the present value of the remaining payments. There are 120 payments in all. After you make 48 of them (the first four years), you have 72 to go. By now, it should be easy for you to verify that the present value of $282.10 per month for 72 months at 0.6875 percent per month is just under $16,000, so you still have a long way to go.

Of course, it is possible to rack up much larger debts. According to the Association of American Medical Colleges, medical students who borrowed to attend medical school and graduated in 2005 had an average student loan balance of $120,280. Ouch! How long will it take the average student to pay off her medical school loans?

SPREADSHEET STRATEGIES

Loan Amortization Using a Spreadsheet

Loan amortization is a common spreadsheet application. To illustrate, we will set up the problem that we examined earlier: a five-year, $5,000, 9 percent loan with constant payments. Our spreadsheet looks like this:

	A	B	C	D	E	F	G	H
1								
2				Using a spreadsheet to amortize a loan				
3								
4			Loan amount:	$5,000				
5			Interest rate:	0.09				
6			Loan term:	5				
7			Loan payment:	**$1,285.46**				
8				Note: Payment is calculated using PMT(rate,nper,-pv,fv).				
9			*Amortization table:*					
10								
11		Year	Beginning	Total	Interest	Principal	Ending	
12			Balance	Payment	Paid	Paid	Balance	
13		1	$5,000.00	$1,285.46	$450.00	$835.46	$4,164.54	
14		2	4,164.54	1,285.46	374.81	910.65	3,253.88	
15		3	3,253.88	1,285.46	292.85	992.61	2,261.27	
16		4	2,261.27	1,285.46	203.51	1,081.95	1,179.32	
17		5	1,179.32	1,285.46	106.14	1,179.32	0.00	
18		Totals		6,427.31	1,427.31	5,000.00		
19								
20			*Formulas in the amortization table:*					
21								
22		Year	Beginning	Total	Interest	Principal	Ending	
23			Balance	Payment	Paid	Paid	Balance	
24		1	=+D4	=$D$7	=+$D$5*C13	=+D13-E13	=+C13-F13	
25		2	=+G13	=$D$7	=+$D$5*C14	=+D14-E14	=+C14-F14	
26		3	=+G14	=$D$7	=+$D$5*C15	=+D15-E15	=+C15-F15	
27		4	=+G15	=$D$7	=+$D$5*C16	=+D16-E16	=+C16-F16	
28		5	=+G16	=$D$7	=+$D$5*C17	=+D17-E17	=+C17-F17	
29								
30		Note: Totals in the amortization table are calculated using the SUM formula.						
31								

Let's say she makes a monthly payment of $1,000, and the loan has an interest rate of 7 percent per year, or .5833 percent per month. See if you agree that it will take 208 months, or just over 17 years, to pay off the loan. Maybe MD really stands for "mucho debt"!

Concept Questions

6.4a What is a pure discount loan? An interest-only loan?

6.4b What does it mean to amortize a loan?

6.4c What is a balloon payment? How do you determine its value?

Summary and Conclusions

6.5

This chapter rounded out your understanding of fundamental concepts related to the time value of money and discounted cash flow valuation. Several important topics were covered:

1. There are two ways of calculating present and future values when there are multiple cash flows. Both approaches are straightforward extensions of our earlier analysis of single cash flows.

2. A series of constant cash flows that arrive or are paid at the end of each period is called an ordinary annuity, and we described some useful shortcuts for determining the present and future values of annuities.

3. Interest rates can be quoted in a variety of ways. For financial decisions, it is important that any rates being compared be first converted to effective rates. The relationship between a quoted rate, such as an annual percentage rate (APR), and an effective annual rate (EAR) is given by:

$$EAR = [1 + (Quoted\ rate/m)]^m - 1$$

where m is the number of times during the year the money is compounded or, equivalently, the number of payments during the year.

4. Many loans are annuities. The process of providing for a loan to be paid off gradually is called amortizing the loan, and we discussed how amortization schedules are prepared and interpreted.

The principles developed in this chapter will figure prominently in the chapters to come. The reason for this is that most investments, whether they involve real assets or financial assets, can be analyzed using the discounted cash flow (DCF) approach. As a result, the DCF approach is broadly applicable and widely used in practice. For example, the next two chapters show how to value bonds and stocks using an extension of the techniques presented in this chapter. Before going on, therefore, you might want to do some of the problems that follow.

Visit us at www.mhhe.com/rwj

CHAPTER REVIEW AND SELF-TEST PROBLEMS

6.1 **Present Values with Multiple Cash Flows** A first-round draft choice quarterback has been signed to a three-year, $25 million contract. The details provide for an immediate cash bonus of $2 million. The player is to receive $5 million in salary at the end of the first year, $8 million the next, and $10 million at the end of the last year. Assuming a 15 percent discount rate, is this package worth $25 million? If not, how much is it worth?

6.2 **Future Value with Multiple Cash Flows** You plan to make a series of deposits in an individual retirement account. You will deposit $1,000 today, $2,000 in two years, and $2,000 in five years. If you withdraw $1,500 in three years and $1,000 in seven years, assuming no withdrawal penalties, how much will you have after eight years if the interest rate is 7 percent? What is the present value of these cash flows?

6.3 **Annuity Present Value** You are looking into an investment that will pay you $12,000 per year for the next 10 years. If you require a 15 percent return, what is the most you would pay for this investment?

6.4   APR versus EAR   The going rate on student loans is quoted as 8 percent APR. The terms of the loans call for monthly payments. What is the effective annual rate (EAR) on such a student loan?

6.5   It's the Principal That Matters   Suppose you borrow $10,000. You are going to repay the loan by making equal annual payments for five years. The interest rate on the loan is 14 percent per year. Prepare an amortization schedule for the loan. How much interest will you pay over the life of the loan?

6.6   Just a Little Bit Each Month   You've recently finished your MBA at the Darnit School. Naturally, you must purchase a new BMW immediately. The car costs about $21,000. The bank quotes an interest rate of 15 percent APR for a 72-month loan with a 10 percent down payment. You plan on trading the car in for a new one in two years. What will your monthly payment be? What is the effective interest rate on the loan? What will the loan balance be when you trade the car in?

ANSWERS TO CHAPTER REVIEW AND SELF-TEST PROBLEMS

6.1   Obviously, the package is not worth $25 million because the payments are spread out over three years. The bonus is paid today, so it's worth $2 million. The present values for the three subsequent salary payments are:

$$(\$5/1.15) + (8/1.15^2) + (10/1.15^3) = (\$5/1.15) + (8/1.32) + (10/1.52)$$
$$= \$16.9721 \text{ million}$$

The package is worth a total of $18.9721 million.

6.2   We will calculate the future values for each of the cash flows separately and then add them up. Notice that we treat the withdrawals as negative cash flows:

$$
\begin{array}{rll}
\$1,000 \times 1.07^8 = & \$1,000 \times 1.7812 = & \$\ 1,718.19 \\
\$2,000 \times 1.07^6 = & \$2,000 \times 1.5007 = & 3,001.46 \\
-\$1,500 \times 1.07^5 = & -\$1,500 \times 1.4026 = & -2,103.83 \\
\$2,000 \times 1.07^3 = & \$2,000 \times 1.2250 = & 2,450.09 \\
-\$1,000 \times 1.07^1 = & -\$1,000 \times 1.0700 = & -1,070.00 \\
\text{Total future value} & & = \$\ 3,995.91
\end{array}
$$

This value includes a small rounding error.

To calculate the present value, we could discount each cash flow back to the present or we could discount back a single year at a time. However, because we already know that the future value in eight years is $3,995.91, the easy way to get the PV is just to discount this amount back eight years:

$$
\begin{aligned}
\text{Present value} &= \$3,995.91/1.07^8 \\
&= \$3,995.91/1.7182 \\
&= \$2,325.64
\end{aligned}
$$

We again ignore a small rounding error. For practice, you can verify that this is what you get if you discount each cash flow back separately.

6.3   The most you would be willing to pay is the present value of $12,000 per year for 10 years at a 15 percent discount rate. The cash flows here are in ordinary annuity form, so the relevant present value factor is:

$$\text{Annuity present value factor} = (1 - \text{Present value factor})/r$$
$$= [1 - (1/1.15^{10})]/.15$$
$$= (1 - .2472)/.15$$
$$= 5.0188$$

The present value of the 10 cash flows is thus:

$$\text{Present value} = \$12,000 \times 5.0188$$
$$= \$60,225$$

This is the most you would pay.

6.4 A rate of 8 percent APR with monthly payments is actually $8\%/12 = .67\%$ per month. The EAR is thus:

$$\text{EAR} = [1 + (.08/12)]^{12} - 1 = 8.30\%$$

6.5 We first need to calculate the annual payment. With a present value of $10,000, an interest rate of 14 percent, and a term of five years, the payment can be determined from:

$$\$10,000 = \text{Payment} \times \{[1 - (1/1.14^5)]/.14\}$$
$$= \text{Payment} \times 3.4331$$

Therefore, the payment is $\$10,000/3.4331 = \$2,912.84$ (actually, it's $2,912.8355; this will create some small rounding errors in the following schedule). We can now prepare the amortization schedule as follows:

Year	Beginning Balance	Total Payment	Interest Paid	Principal Paid	Ending Balance
1	$10,000.00	$ 2,912.84	$1,400.00	$ 1,512.84	$8,487.16
2	8,487.16	2,912.84	1,188.20	1,724.63	6,762.53
3	6,762.53	2,912.84	946.75	1,966.08	4,796.45
4	4,796.45	2,912.84	671.50	2,241.33	2,555.12
5	2,555.12	2,912.84	357.72	2,555.12	0.00
Totals		$14,564.17	$4,564.17	$10,000.00	

6.6 The cash flows on the car loan are in annuity form, so we need to find only the payment. The interest rate is $15\%/12 = 1.25\%$ per month, and there are 72 months. The first thing we need is the annuity factor for 72 periods at 1.25 percent per period:

$$\text{Annuity present value factor} = (1 - \text{Present value factor})/r$$
$$= [1 - (1/1.0125^{72})]/.0125$$
$$= [1 - (1/2.4459)]/.0125$$
$$= (1 - .4088)/.0125$$
$$= 47.2925$$

The present value is the amount we finance. With a 10 percent down payment, we will be borrowing 90 percent of $21,000, or $18,900. To find the payment, we need to solve for C:

$$\$18,900 = C \times \text{Annuity present value factor}$$
$$= C \times 47.2925$$

Rearranging things a bit, we have:

$$C = \$18,900 \times (1/47.2925)$$
$$= \$18,900 \times .02115$$
$$= \$399.64$$

Your payment is just under $400 per month.

The actual interest rate on this loan is 1.25 percent per month. Based on our work in the chapter, we can calculate the effective annual rate as:

$$EAR = (1.0125)^{12} - 1 = 16.08\%$$

The effective rate is about one point higher than the quoted rate.

To determine the loan balance in two years, we could amortize the loan to see what the balance is at that time. This would be fairly tedious to do by hand. Using the information already determined in this problem, we can instead simply calculate the present value of the remaining payments. After two years, we have made 24 payments, so there are $72 - 24 = 48$ payments left. What is the present value of 48 monthly payments of $399.64 at 1.25 percent per month? The relevant annuity factor is:

$$\text{Annuity present value factor} = (1 - \text{Present value factor})/r$$
$$= [1 - (1/1.0125^{48})]/.0125$$
$$= [1 - (1/1.8154)]/.0125$$
$$= (1 - .5509)/.0125$$
$$= 35.9315$$

The present value is thus:

$$\text{Present value} = \$399.64 \times 35.9315 = \$14,359.66$$

You will owe about $14,360 on the loan in two years.

CONCEPTS REVIEW AND CRITICAL THINKING QUESTIONS

1. **Annuity Factors** There are four pieces to an annuity present value. What are they?
2. **Annuity Period** As you increase the length of time involved, what happens to the present value of an annuity? What happens to the future value?
3. **Interest Rates** What happens to the future value of an annuity if you increase the rate r? What happens to the present value?
4. **Present Value** What do you think about the Tri-State Megabucks lottery discussed in the chapter advertising a $500,000 prize when the lump sum option is $250,000? Is it deceptive advertising?
5. **Present Value** If you were an athlete negotiating a contract, would you want a big signing bonus payable immediately and smaller payments in the future, or vice versa? How about looking at it from the team's perspective?
6. **Present Value** Suppose two athletes sign 10-year contracts for $80 million. In one case, we're told that the $80 million will be paid in 10 equal installments. In the other case, we're told that the $80 million will be paid in 10 installments, but the installments will increase by 5 percent per year. Who got the better deal?
7. **APR and EAR** Should lending laws be changed to require lenders to report EARs instead of APRs? Why or why not?

8. **Time Value** On subsidized Stafford loans, a common source of financial aid for college students, interest does not begin to accrue until repayment begins. Who receives a bigger subsidy, a freshman or a senior? Explain. In words, how would you go about valuing the subsidy on a subsidized Stafford loan?

9. **Time Value** Eligibility for a subsidized Stafford loan is based on current financial need. However, both subsidized and unsubsidized Stafford loans are repaid out of future income. Given this, do you see a possible objection to having two types?

10. **Time Value** A viatical settlement is a lump sum of money given to a terminally ill individual in exchange for his life insurance policy. When the insured person dies, the purchaser receives the payout from the life insurance policy. What factors determine the value of the viatical settlement? Do you think such settlements are ethical? Why or why not?

™

QUESTIONS AND PROBLEMS

1. **Present Value and Multiple Cash Flows** Seaborn Co. has identified an investment project with the following cash flows. If the discount rate is 10 percent, what is the present value of these cash flows? What is the present value at 18 percent? At 24 percent?

BASIC
(Questions 1–28)

Year	Cash Flow
1	$1,100
2	720
3	940
4	1,160

2. **Present Value and Multiple Cash Flows** Investment X offers to pay you $7,000 per year for eight years, whereas Investment Y offers to pay you $9,000 per year for five years. Which of these cash flow streams has the higher present value if the discount rate is 5 percent? If the discount rate is 22 percent?

3. **Future Value and Multiple Cash Flows** Paradise, Inc., has identified an investment project with the following cash flows. If the discount rate is 8 percent, what is the future value of these cash flows in year 4? What is the future value at a discount rate of 11 percent? At 24 percent?

Year	Cash Flow
1	$ 700
2	950
3	1,200
4	1,300

4. **Calculating Annuity Present Value** An investment offers $4,600 per year for 15 years, with the first payment occurring one year from now. If the required return is 8 percent, what is the value of the investment? What would the value be if the payments occurred for 40 years? For 75 years? Forever?

5. **Calculating Annuity Cash Flows** If you put up $28,000 today in exchange for a 8.25 percent, 15-year annuity, what will the annual cash flow be?

6. **Calculating Annuity Values** Your company will generate $65,000 in annual revenue each year for the next eight years from a new information database. If the appropriate interest rate is 8.5 percent, what is the present value of the savings?

7. **Calculating Annuity Values** If you deposit $3,000 at the end of each of the next 20 years into an account paying 10.5 percent interest, how much money will you have in the account in 20 years? How much will you have if you make deposits for 40 years?

8. **Calculating Annuity Values** You want to have $80,000 in your savings account 10 years from now, and you're prepared to make equal annual deposits into the account at the end of each year. If the account pays 6.5 percent interest, what amount must you deposit each year?

9. **Calculating Annuity Values** Dinero Bank offers you a $30,000, seven-year term loan at 8 percent annual interest. What will your annual loan payment be?

10. **Calculating Perpetuity Values** The Maybe Pay Life Insurance Co. is trying to sell you an investment policy that will pay you and your heirs $20,000 per year forever. If the required return on this investment is 8 percent, how much will you pay for the policy?

11. **Calculating Perpetuity Values** In the previous problem, suppose a sales associate told you the policy costs $280,000. At what interest rate would this be a fair deal?

12. **Calculating EAR** Find the EAR in each of the following cases:

Stated Rate (APR)	Number of Times Compounded	Effective Rate (EAR)
7%	Quarterly	
18	Monthly	
10	Daily	
14	Infinite	

13. **Calculating APR** Find the APR, or stated rate, in each of the following cases:

Stated Rate (APR)	Number of Times Compounded	Effective Rate (EAR)
	Semiannually	12.2%
	Monthly	9.4
	Weekly	8.6
	Infinite	23.8

14. **Calculating EAR** First National Bank charges 13.1 percent compounded monthly on its business loans. First United Bank charges 13.4 percent compounded semiannually. As a potential borrower, which bank would you go to for a new loan?

15. **Calculating APR** Tarpley Credit Corp. wants to earn an effective annual return on its consumer loans of 14 percent per year. The bank uses daily compounding on its loans. What interest rate is the bank required by law to report to potential borrowers? Explain why this rate is misleading to an uninformed borrower.

16. **Calculating Future Values** What is the future value of $1,400 in 20 years assuming an interest rate of 9.6 percent compounded semiannually?

17. **Calculating Future Values** Corpstein Credit Bank is offering 8.4 percent compounded daily on its savings accounts. If you deposit $6,000 today, how much will you have in the account in 5 years? In 10 years? In 20 years?

18. **Calculating Present Values** An investment will pay you $45,000 in six years. If the appropriate discount rate is 11 percent compounded daily, what is the present value?

19. **EAR versus APR** Big Dom's Pawn Shop charges an interest rate of 25 percent per month on loans to its customers. Like all lenders, Big Dom must report an APR to consumers. What rate should the shop report? What is the effective annual rate?

20. **Calculating Loan Payments** You want to buy a new sports coupe for $61,800, and the finance office at the dealership has quoted you a 7.4 percent APR loan for 60 months to buy the car. What will your monthly payments be? What is the effective annual rate on this loan?

21. **Calculating Number of Periods** One of your customers is delinquent on his accounts payable balance. You've mutually agreed to a repayment schedule of $300 per month. You will charge .9 percent per month interest on the overdue balance. If the current balance is $17,000, how long will it take for the account to be paid off?

22. **Calculating EAR** Friendly's Quick Loans, Inc., offers you "three for four or I knock on your door." This means you get $3 today and repay $4 when you get your paycheck in one week (or else). What's the effective annual return Friendly's earns on this lending business? If you were brave enough to ask, what APR would Friendly's say you were paying?

23. **Valuing Perpetuities** Live Forever Life Insurance Co. is selling a perpetuity contract that pays $1,200 monthly. The contract currently sells for $63,000. What is the monthly return on this investment vehicle? What is the APR? The effective annual return?

24. **Calculating Annuity Future Values** You are planning to make monthly deposits of $250 into a retirement account that pays 10 percent interest compounded monthly. If your first deposit will be made one month from now, how large will your retirement account be in 30 years?

25. **Calculating Annuity Future Values** In the previous problem, suppose you make $3,000 annual deposits into the same retirement account. How large will your account balance be in 30 years?

26. **Calculating Annuity Present Values** Beginning three months from now, you want to be able to withdraw $1,500 each quarter from your bank account to cover college expenses over the next four years. If the account pays .75 percent interest per quarter, how much do you need to have in your bank account today to meet your expense needs over the next four years?

27. **Discounted Cash Flow Analysis** If the appropriate discount rate for the following cash flows is 11 percent compounded quarterly, what is the present value of the cash flows?

Year	Cash Flow
1	$ 900
2	850
3	0
4	1,140

28. **Discounted Cash Flow Analysis** If the appropriate discount rate for the following cash flows is 8.45 percent per year, what is the present value of the cash flows?

Year	Cash Flow
1	$2,800
2	0
3	5,600
4	1,940

Visit us at www.mhhe.com/rwj

INTERMEDIATE
(Questions 29–56)

29. Simple Interest versus Compound Interest First Simple Bank pays 6 percent simple interest on its investment accounts. If First Complex Bank pays interest on its accounts compounded annually, what rate should the bank set if it wants to match First Simple Bank over an investment horizon of 10 years?

30. Calculating EAR You are looking at an investment that has an effective annual rate of 18 percent. What is the effective semiannual return? The effective quarterly return? The effective monthly return?

31. Calculating Interest Expense You receive a credit card application from Shady Banks Savings and Loan offering an introductory rate of 2.5 percent per year, compounded monthly for the first six months, increasing thereafter to 17 percent compounded monthly. Assuming you transfer the $5,000 balance from your existing credit card and make no subsequent payments, how much interest will you owe at the end of the first year?

32. Calculating Annuities You are planning to save for retirement over the next 30 years. To do this, you will invest $600 a month in a stock account and $300 a month in a bond account. The return of the stock account is expected to be 12 percent, and the bond account will pay 7 percent. When you retire, you will combine your money into an account with a 9 percent return. How much can you withdraw each month from your account assuming a 25-year withdrawal period?

33. Calculating Future Values You have an investment that will pay you 1.08 percent per month. How much will you have per dollar invested in one year? In two years?

34. Calculating Annuity Payments You want to be a millionaire when you retire in 40 years. How much do you have to save each month if you can earn an 11 percent annual return? How much do you have to save if you wait 10 years before you begin your deposits? 20 years?

35. Calculating Rates of Return Suppose an investment offers to quadruple your money in 12 months (don't believe it). What rate of return per quarter are you being offered?

36. Comparing Cash Flow Streams You've just joined the investment banking firm of Dewey, Cheatum, and Howe. They've offered you two different salary arrangements. You can have $90,000 per year for the next two years, or you can have $65,000 per year for the next two years, along with a $45,000 signing bonus today. The bonus is paid immediately, and the salary is paid at the end of each year. If the interest rate is 10 percent compounded monthly, which do you prefer?

37. Growing Annuity You have just won the lottery and will receive $1,000,000 in one year. You will receive payments for 25 years, which will increase 5 percent per year. If the appropriate discount rate is 9 percent, what is the present value of your winnings?

38. Growing Annuity Your job pays you only once a year for all the work you did over the previous 12 months. Today, December 31, you just received your salary of $50,000 and you plan to spend all of it. However, you want to start saving for retirement beginning next year. You have decided that one year from today you will begin depositing 2 percent of your annual salary in an account that will earn 10 percent per year. Your salary will increase at 4 percent per year throughout your career. How much money will you have on the date of your retirement 40 years from today?

39. Present Value and Interest Rates What is the relationship between the value of an annuity and the level of interest rates? Suppose you just bought a 10-year annuity of $7,000 per year at the current interest rate of 10 percent per year. What

happens to the value of your investment if interest rates suddenly drop to 5 percent? What if interest rates suddenly rise to 15 percent?

40.   Calculating the Number of Payments You're prepared to make monthly payments of $225, beginning at the end of this month, into an account that pays 9 percent interest compounded monthly. How many payments will you have made when your account balance reaches $20,000?

41.   Calculating Annuity Present Values You want to borrow $55,000 from your local bank to buy a new sailboat. You can afford to make monthly payments of $1,120, but no more. Assuming monthly compounding, what is the highest rate you can afford on a 60-month APR loan?

42.   Calculating Loan Payments You need a 30-year, fixed-rate mortgage to buy a new home for $220,000. Your mortgage bank will lend you the money at a 6.8 percent APR for this 360-month loan. However, you can afford monthly payments of only $1,100, so you offer to pay off any remaining loan balance at the end of the loan in the form of a single balloon payment. How large will this balloon payment have to be for you to keep your monthly payments at $1,100?

43.   Present and Future Values The present value of the following cash flow stream is $6,785 when discounted at 10 percent annually. What is the value of the missing cash flow?

Year	Cash Flow
1	$1,500
2	?
3	1,800
4	2,400

44.   Calculating Present Values You just won the TVM Lottery. You will receive $1 million today plus another 10 annual payments that increase by $400,000 per year. Thus, in one year, you receive $1.4 million. In two years you get $1.8 million, and so on. If the appropriate interest rate is 9 percent, what is the present value of your winnings?

45.   EAR versus APR You have just purchased a new warehouse. To finance the purchase, you've arranged for a 30-year mortgage loan for 80 percent of the $2,400,000 purchase price. The monthly payment on this loan will be $13,000. What is the APR on this loan? The EAR?

46.   Present Value and Break-Even Interest Consider a firm with a contract to sell an asset for $145,000 three years from now. The asset costs $94,000 to produce today. Given a relevant discount rate on this asset of 13 percent per year, will the firm make a profit on this asset? At what rate does the firm just break even?

47.   Present Value and Multiple Cash Flows What is the present value of $2,000 per year, at a discount rate of 10 percent, if the first payment is received 9 years from now and the last payment is received 25 years from now?

48.   Variable Interest Rates A 15-year annuity pays $1,500 per month, and payments are made at the end of each month. If the interest rate is 13 percent compounded monthly for the first seven years, and 10 percent compounded monthly thereafter, what is the present value of the annuity?

49.   Comparing Cash Flow Streams You have your choice of two investment accounts. Investment A is a 15-year annuity that features end-of-month $1,000

payments and has an interest rate of 9.5 percent compounded monthly. Investment B is a 9 percent continuously compounded lump sum investment, also good for 15 years. How much money would you need to invest in B today for it to be worth as much as investment A 15 years from now?

50. **Calculating Present Value of a Perpetuity** Given an interest rate of 5.7 percent per year, what is the value at date $t = 7$ of a perpetual stream of $5,000 payments that begins at date $t = 15$?

51. **Calculating EAR** A local finance company quotes a 15 percent interest rate on one-year loans. So, if you borrow $20,000, the interest for the year will be $3,000. Because you must repay a total of $23,000 in one year, the finance company requires you to pay $23,000/12, or $1,916.67, per month over the next 12 months. Is this a 15 percent loan? What rate would legally have to be quoted? What is the effective annual rate?

52. **Calculating Present Values** A 5-year annuity of ten $6,000 semiannual payments will begin 9 years from now, with the first payment coming 9.5 years from now. If the discount rate is 10 percent compounded monthly, what is the value of this annuity five years from now? What is the value three years from now? What is the current value of the annuity?

53. **Calculating Annuities Due** As discussed in the text, an ordinary annuity assumes equal payments at the end of each period over the life of the annuity. An *annuity due* is the same thing except the payments occur at the beginning of each period instead. Thus, a three-year annual annuity due would have periodic payment cash flows occurring at years 0, 1, and 2, whereas a three-year annual ordinary annuity would have periodic payment cash flows occurring at years 1, 2, and 3.
   a. At a 9.5 percent annual discount rate, find the present value of an eight-year ordinary annuity contract of $950 payments.
   b. Find the present value of the same contract if it is an annuity due.

54. **Calculating Annuities Due** You want to buy a new sports car from Muscle Motors for $61,000. The contract is in the form of a 60-month annuity due at an 8.15 percent APR. What will your monthly payment be?

55. **Amortization with Equal Payments** Prepare an amortization schedule for a five-year loan of $36,000. The interest rate is 9 percent per year, and the loan calls for equal annual payments. How much interest is paid in the third year? How much total interest is paid over the life of the loan?

56. **Amortization with Equal Principal Payments** Rework Problem 55 assuming that the loan agreement calls for a principal reduction of $7,200 every year instead of equal annual payments.

CHALLENGE
(Questions 57–78)

57. **Calculating Annuity Values** Bilbo Baggins wants to save money to meet three objectives. First, he would like to be able to retire 30 years from now with retirement income of $20,000 per month for 20 years, with the first payment received 30 years and 1 month from now. Second, he would like to purchase a cabin in Rivendell in 10 years at an estimated cost of $325,000. Third, after he passes on at the end of the 20 years of withdrawals, he would like to leave an inheritance of $750,000 to his nephew Frodo. He can afford to save $2,000 per month for the next 10 years. If he can earn an 11 percent EAR before he retires and an 8 percent EAR after he retires, how much will he have to save each month in years 11 through 30?

58. **Calculating Annuity Values** After deciding to buy a new car, you can either lease the car or purchase it on a three-year loan. The car you wish to buy costs $28,000.

The dealer has a special leasing arrangement where you pay $1 today and $380 per month for the next three years. If you purchase the car, you will pay it off in monthly payments over the next three years at an 8 percent APR. You believe you will be able to sell the car for $15,000 in three years. Should you buy or lease the car? What break-even resale price in three years would make you indifferent between buying and leasing?

59. **Calculating Annuity Values** An All-Pro defensive lineman is in contract negotiations. The team has offered the following salary structure:

Time	Salary
0	$8,000,000
1	$4,000,000
2	$4,800,000
3	$5,700,000
4	$6,400,000
5	$7,000,000
6	$7,500,000

All salaries are to be paid in lump sums. The player has asked you as his agent to renegotiate the terms. He wants a $9 million signing bonus payable today and a contract value increase of $750,000. He also wants an equal salary paid every three months, with the first paycheck three months from now. If the interest rate is 5.5 percent compounded daily, what is the amount of his quarterly check? Assume 365 days in a year.

60. **Discount Interest Loans** This question illustrates what is known as *discount interest*. Imagine you are discussing a loan with a somewhat unscrupulous lender. You want to borrow $20,000 for one year. The interest rate is 14 percent. You and the lender agree that the interest on the loan will be .14 × $20,000 = $2,800. So the lender deducts this interest amount from the loan up front and gives you $17,200. In this case, we say that the discount is $2,800. What's wrong here?

61. **Calculating Annuity Values** You are serving on a jury. A plaintiff is suing the city for injuries sustained after a freak street sweeper accident. In the trial, doctors testified that it will be five years before the plaintiff is able to return to work. The jury has already decided in favor of the plaintiff. You are the foreperson of the jury and propose that the jury give the plaintiff an award to cover the following: (a) The present value of two years' back pay. The plaintiff's annual salary for the last two years would have been $44,000 and $46,000, respectively. (b) The present value of five years' future salary. You assume the salary will be $49,000 per year. (c) $100,000 for pain and suffering. (d) $20,000 for court costs. Assume that the salary payments are equal amounts paid at the end of each month. If the interest rate you choose is a 9 percent EAR, what is the size of the settlement? If you were the plaintiff, would you like to see a higher or lower interest rate?

62. **Calculating EAR with Points** You are looking at a one-year loan of $10,000. The interest rate is quoted as 9 percent plus three points. A *point* on a loan is simply 1 percent (one percentage point) of the loan amount. Quotes similar to this one are common with home mortgages. The interest rate quotation in this example requires the borrower to pay three points to the lender up front and repay the loan later with 9 percent interest. What rate would you actually be paying here?

63.   Calculating EAR with Points The interest rate on a one-year loan is quoted as 12 percent plus two points (see the previous problem). What is the EAR? Is your answer affected by the loan amount?

64.   EAR versus APR Two banks in the area offer 30-year, $220,000 mortgages at 7.2 percent and charge a $1,500 loan application fee. However, the application fee charged by Insecurity Bank and Trust is refundable if the loan application is denied, whereas that charged by I.M. Greedy and Sons Mortgage Bank is not. The current disclosure law requires that any fees that will be refunded if the applicant is rejected be included in calculating the APR, but this is not required with nonrefundable fees (presumably because refundable fees are part of the loan rather than a fee). What are the EARs on these two loans? What are the APRs?

65.   Calculating EAR with Add-On Interest This problem illustrates a deceptive way of quoting interest rates called *add-on interest*. Imagine that you see an advertisement for Crazy Judy's Stereo City that reads something like this: "$1,000 Instant Credit! 13% Simple Interest! Three Years to Pay! Low, Low Monthly Payments!" You're not exactly sure what all this means and somebody has spilled ink over the APR on the loan contract, so you ask the manager for clarification.

Judy explains that if you borrow $1,000 for three years at 13 percent interest, in three years you will owe:

$$\$1,000 \times 1.13^3 = \$1,000 \times 1.42290 = \$1,442.90$$

Now, Judy recognizes that coming up with $1,442.90 all at once might be a strain, so she lets you make "low, low monthly payments" of $1,442.90/36 = $40.08 per month, even though this is extra bookkeeping work for her.

Is this a 13 percent loan? Why or why not? What is the APR on this loan? What is the EAR? Why do you think this is called add-on interest?

66.   Calculating Annuity Payments This is a classic retirement problem. A time line will help in solving it. Your friend is celebrating her 35th birthday today and wants to start saving for her anticipated retirement at age 65. She wants to be able to withdraw $90,000 from her savings account on each birthday for 20 years following her retirement; the first withdrawal will be on her 66th birthday. Your friend intends to invest her money in the local credit union, which offers 8 percent interest per year. She wants to make equal annual payments on each birthday into the account established at the credit union for her retirement fund.

a. If she starts making these deposits on her 36th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals at retirement?

b. Suppose your friend has just inherited a large sum of money. Rather than making equal annual payments, she has decided to make one lump sum payment on her 35th birthday to cover her retirement needs. What amount does she have to deposit?

c. Suppose your friend's employer will contribute $1,500 to the account every year as part of the company's profit-sharing plan. In addition, your friend expects a $25,000 distribution from a family trust fund on her 55th birthday, which she will also put into the retirement account. What amount must she deposit annually now to be able to make the desired withdrawals at retirement?

67.   Calculating the Number of Periods Your Christmas ski vacation was great, but it unfortunately ran a bit over budget. All is not lost: You just received an offer in the mail to transfer your $10,000 balance from your current credit card, which charges an annual rate of 18.2 percent, to a new credit card charging a rate of 8.2 percent.

How much faster could you pay the loan off by making your planned monthly payments of $200 with the new card? What if there was a 2 percent fee charged on any balances transferred?

68. **Future Value and Multiple Cash Flows** An insurance company is offering a new policy to its customers. Typically, the policy is bought by a parent or grandparent for a child at the child's birth. The details of the policy are as follows: The purchaser (say, the parent) makes the following six payments to the insurance company:

First birthday:	$ 800
Second birthday:	$ 800
Third birthday:	$ 900
Fourth birthday:	$ 900
Fifth birthday:	$1,000
Sixth birthday:	$1,000

After the child's sixth birthday, no more payments are made. When the child reaches age 65, he or she receives $350,000. If the relevant interest rate is 11 percent for the first six years and 7 percent for all subsequent years, is the policy worth buying?

69. **Calculating a Balloon Payment** You have just arranged for a $450,000 mortgage to finance the purchase of a large tract of land. The mortgage has an 8.5 percent APR, and it calls for monthly payments over the next 30 years. However, the loan has an eight-year balloon payment, meaning that the loan must be paid off then. How big will the balloon payment be?

70. **Calculating Interest Rates** A financial planning service offers a college savings program. The plan calls for you to make six annual payments of $5,000 each, with the first payment occurring today, your child's 12th birthday. Beginning on your child's 18th birthday, the plan will provide $15,000 per year for four years. What return is this investment offering?

71. **Break-Even Investment Returns** Your financial planner offers you two different investment plans. Plan X is a $15,000 annual perpetuity. Plan Y is a 10-year, $20,000 annual annuity. Both plans will make their first payment one year from today. At what discount rate would you be indifferent between these two plans?

72. **Perpetual Cash Flows** What is the value of an investment that pays $7,500 every *other* year forever, if the first payment occurs one year from today and the discount rate is 11 percent compounded daily? What is the value today if the first payment occurs four years from today?

73. **Ordinary Annuities and Annuities Due** As discussed in the text, an annuity due is identical to an ordinary annuity except that the periodic payments occur at the beginning of each period and not at the end of the period. Show that the relationship between the value of an ordinary annuity and the value of an otherwise equivalent annuity due is:

Annuity due value = Ordinary annuity value $\times$ (1 + *r*)

Show this for both present and future values.

74. **Calculating Growing Annuities** You have 30 years left until retirement and want to retire with $1 million. Your salary is paid annually, and you will receive $55,000 at the end of the current year. Your salary will increase at 3 percent per year, and

you can earn a 10 percent return on the money you invest. If you save a constant percentage of your salary, what percentage of your salary must you save each year?

75. **Calculating EAR** A check-cashing store is in the business of making personal loans to walk-up customers. The store makes only one-week loans at 8 percent interest per week.

   a. What APR must the store report to its customers? What EAR are customers actually paying?

   b. Now suppose the store makes one-week loans at 8 percent discount interest per week (see Problem 60). What's the APR now? The EAR?

   c. The check-cashing store also makes one-month add-on interest loans at 8 percent discount interest per week. Thus if you borrow $100 for one month (four weeks), the interest will be ($100 $\times$ 1.08^4) $-$ 100 = $31.08. Because this is discount interest, your net loan proceeds today will be $68.92. You must then repay the store $100 at the end of the month. To help you out, though, the store lets you pay off this $100 in installments of $25 per week. What is the APR of this loan? What is the EAR?

76. **Present Value of a Growing Perpetuity** What is the equation for the present value of a growing perpetuity with a payment of *C* one period from today if the payments grow by *C* each period?

77. **Rule of 72** Earlier, we discussed the Rule of 72, a useful approximation for many interest rates and periods for the time it takes a lump sum to double in value. For a 10 percent interest rate, show that the "Rule of 73" is slightly better. For what rate is the Rule of 72 exact? (*Hint:* Use the Solver function in Excel.)

78. **Rule of 69.3** A corollary to the Rule of 72 is the Rule of 69.3. The Rule of 69.3 is exactly correct except for rounding when interest rates are compounded continuously. Prove the Rule of 69.3 for continuously compounded interest.

WEB EXERCISES

6.1 **Annuity Future Value** The St. Louis Federal Reserve Board has files listing historical interest rates on its Web site: www.stls.frb.org. Find the link for "FRED II," then "Interest Rates." You will find listings for Moody's Seasoned Aaa Corporate Bond Yield and Moody's Seasoned Baa Corporate Bond Yield. (These rates are discussed in the next chapter.) If you invest $2,000 per year for the next 40 years at the most recent Aaa yield, how much will you have? What if you invest the same amount at the Baa yield?

6.2 **Loan Payments** Finding the time necessary until you pay off a loan is simple if you make equal payments each month. However, when paying off credit cards many individuals make only the minimum monthly payment, which is generally $10 or 2 percent to 3 percent of the balance, whichever is greater. You can find a credit card calculator at www.fincalc.com. You currently owe $10,000 on a credit card with a 17 percent interest rate and a minimum payment of $10 or 2 percent of your balance, whichever is greater. How soon will you pay off this debt if you make the minimum payment each month? How much total interest will you pay?

6.3 **Annuity Payments** Go to www.moneychimp.com. Use the calculator to solve this problem. If you have $1,500,000 when you retire and want to withdraw an equal amount for the next 30 years, how much can you withdraw each year if you earn 7 percent? What if you earn 9 percent?

6.4 **Annuity Payments** The St. Louis Federal Reserve Board has files listing historical interest rates on its Web site: www.stls.frb.org. Find the link for "FRED II," then "Interest Rates." You will find a listing for the Bank Prime Loan Rate. The file lists the monthly prime rates since January 1949 (1949.01). What is the most recent prime rate? What is the highest prime rate over this period? If you bought a house for $150,000 at the current prime rate on a 30-year mortgage with monthly payments, how much are your payments? If you had purchased the house at the same price when the prime rate was its highest, what would your monthly payments have been?

6.5 **Loan Amortization** You can find a calculator that will prepare a loan amortization table at www.hsh.com. You want to buy a home for $200,000 on a 30-year mortgage with monthly payments at the rate quoted on the site. What percentage of your first month's payment is principal? What percentage of your last month's payment is principal? What is the total interest paid on the loan?

The MBA Decision

Ben Bates graduated from college six years ago with a finance undergraduate degree. Although he is satisfied with his current job, his goal is to become an investment banker. He feels that an MBA degree would allow him to achieve this goal. After examining schools, he has narrowed his choice to either Wilton University or Mount Perry College. Although internships are encouraged by both schools, to get class credit for the internship, no salary can be paid. Other than internships, neither school will allow its students to work while enrolled in its MBA program.

Ben currently works at the money management firm of Dewey and Louis. His annual salary at the firm is $50,000 per year, and his salary is expected to increase at 3 percent per year until retirement. He is currently 28 years old and expects to work for 35 more years. His current job includes a fully paid health insurance plan, and his current average tax rate is 26 percent. Ben has a savings account with enough money to cover the entire cost of his MBA program.

The Ritter College of Business at Wilton University is one of the top MBA programs in the country. The MBA degree requires two years of full-time enrollment at the university. The annual tuition is $60,000, payable at the beginning of each school year. Books and other supplies are estimated to cost $2,500 per year. Ben expects that after graduation from Wilton, he will receive a job offer for about $95,000 per year, with a $15,000 signing bonus. The salary at this job will increase at 4 percent per year. Because of the higher salary, his average income tax rate will increase to 31 percent.

The Bradley School of Business at Mount Perry College began its MBA program 16 years ago. The Bradley School is smaller and less well known than the Ritter College. Bradley offers an accelerated one-year program, with a tuition cost of $75,000 to be paid upon matriculation. Books and other supplies for the program are expected to cost $3,500. Ben thinks that he will receive an offer of $78,000 per year upon graduation, with a $10,000 signing bonus. The salary at this job will increase at 3.5 percent per year. His average tax rate at this level of income will be 29 percent.

Both schools offer a health insurance plan that will cost $3,000 per year, payable at the beginning of the year. Ben also estimates that room and board expenses will cost $20,000 per year at both schools. The appropriate discount rate is 6.5 percent.

1. How does Ben's age affect his decision to get an MBA?

2. What other, perhaps nonquantifiable, factors affect Ben's decision to get an MBA?

3. Assuming all salaries are paid at the end of each year, what is the best option for Ben from a strictly financial standpoint?

4. Ben believes that the appropriate analysis is to calculate the future value of each option. How would you evaluate this statement?

5. What initial salary would Ben need to receive to make him indifferent between attending Wilton University and staying in his current position?

6. Suppose, instead of being able to pay cash for his MBA, Ben must borrow the money. The current borrowing rate is 5.4 percent. How would this affect his decision?